Ch 13 Vector valued functions

generalization of parametric equations.

x, y, z usual coords

t additional parameter

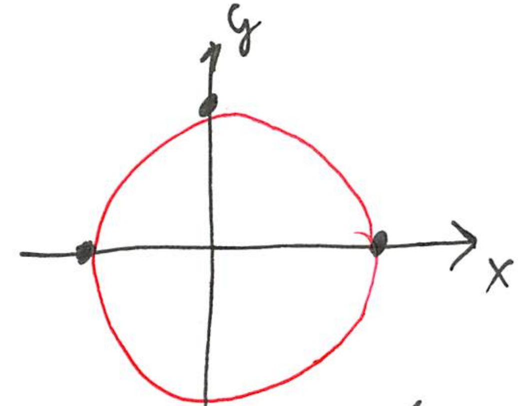
We give x, y, z as functions of t

e.g. line in 3D

$$\begin{cases} x = 2t+1 \\ y = t \\ z = -3t-7 \end{cases} \quad t \in \mathbb{R}.$$

e.g. $x = \cos t$, $y = \sin t$

t	x	y
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0



parametric eqns for unit circle.

e.g. if $y = f(x)$, we can get parametric eqns for the graph by putting $x = t$, $y = f(t)$

more about parametric eqns in §10.1

Vector-valued function (VVF) is a function with

input t

output $\vec{r}(t)$

e.g. $\vec{r}(t) = \langle 2t+1, t, -3t-7 \rangle$

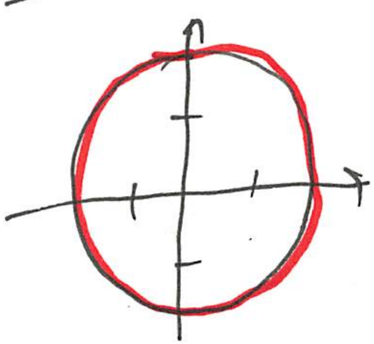
VVF is really just a package of functions, one for each coordinate.

Graphing a VVF means plotting the points that lie at the tips of the vectors.

e.g $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$

<u>2D example</u>

$x = 2\cos t$
$y = 2\sin t$

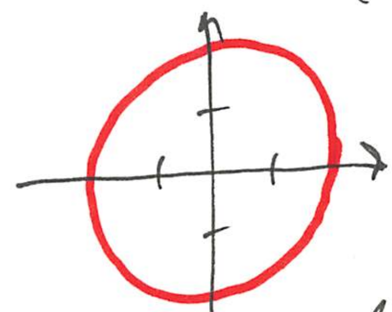


graph is a circle of radius 2.

VVF is more than just the shape of the graph. It tells us how we move along that shape. Can think of t as representing time.

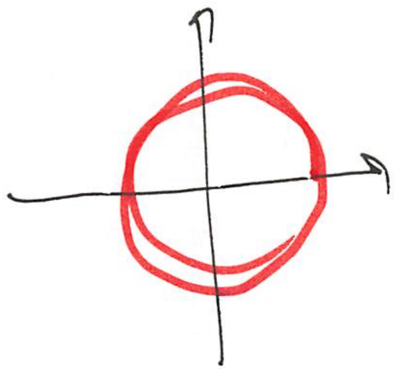
Then $\vec{r}(t)$ is like a trajectory of a particle

e.g $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$



uniform circular motion; move at constant speed around the circle.

e.g. $\vec{r}(t) = \langle 2\cos(t^3), 2\sin(t^3) \rangle$
still get circle of radius 2 at origin.
But this time speed is not constant.

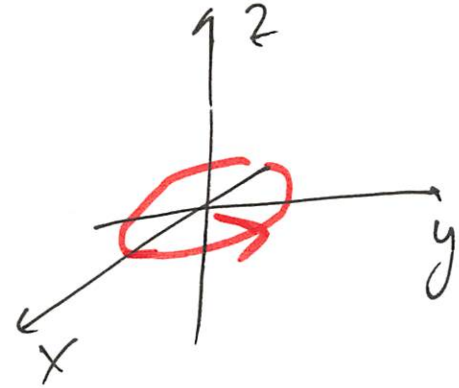


e.g. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$
What does graph look like?
Think of as
$\vec{r}(t) = \vec{f}_1(t) + \vec{f}_2(t)$
where

$\vec{f}_1(t) = \langle \cos t, \sin t, 0 \rangle$
$\vec{f}_2(t) = \langle 0, 0, t \rangle$

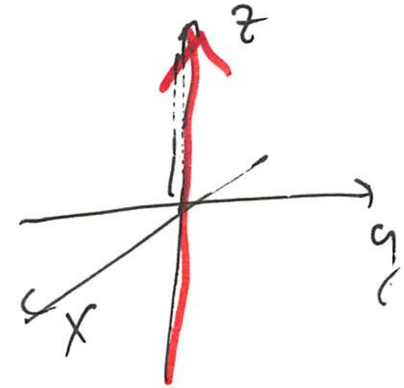
$\vec{f}_1(t)$:

uniform circular motion in xy plane.



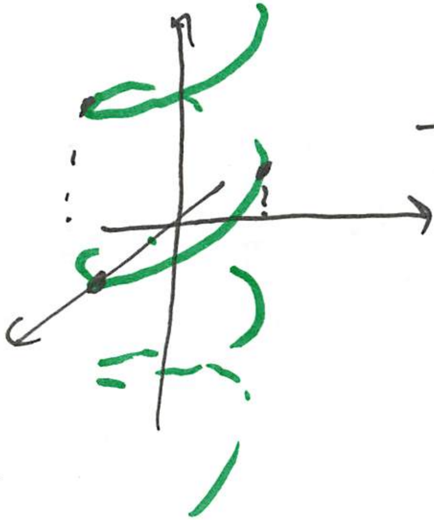
$\vec{f}_2(t)$:

along z axis
uniform speed, up z axis.

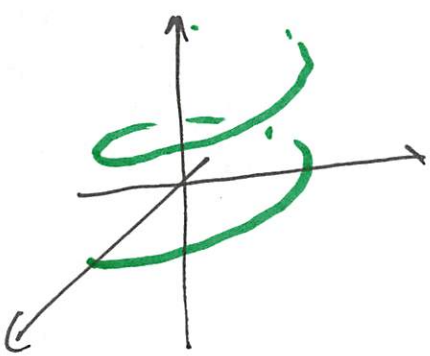


To get actual motion, we superimpose.

Helix

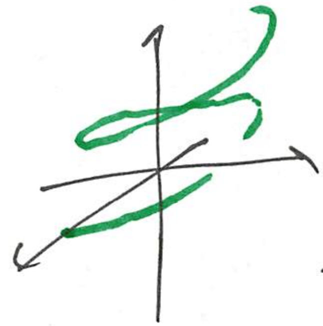


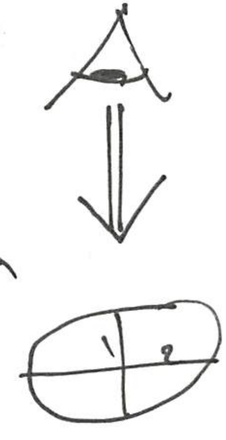
e.g. $\vec{r}(t) = \langle 2\cos t, 2\sin t, t \rangle$

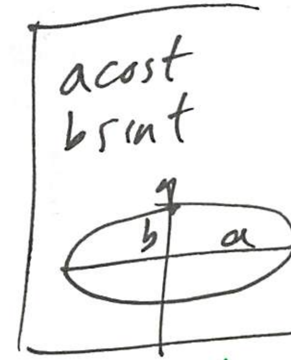


e.g. $\vec{r}(t) = \langle \underbrace{2\cos t, \sin t}_{\text{ellipse.}}, t \rangle$

elliptical helix



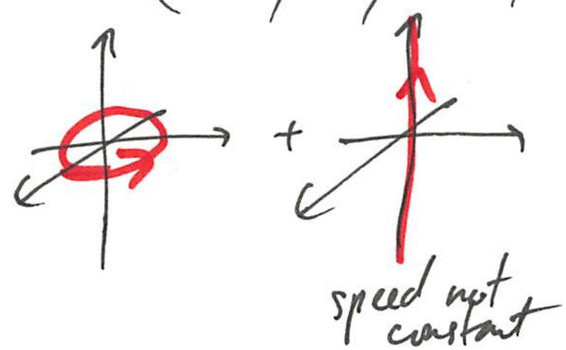




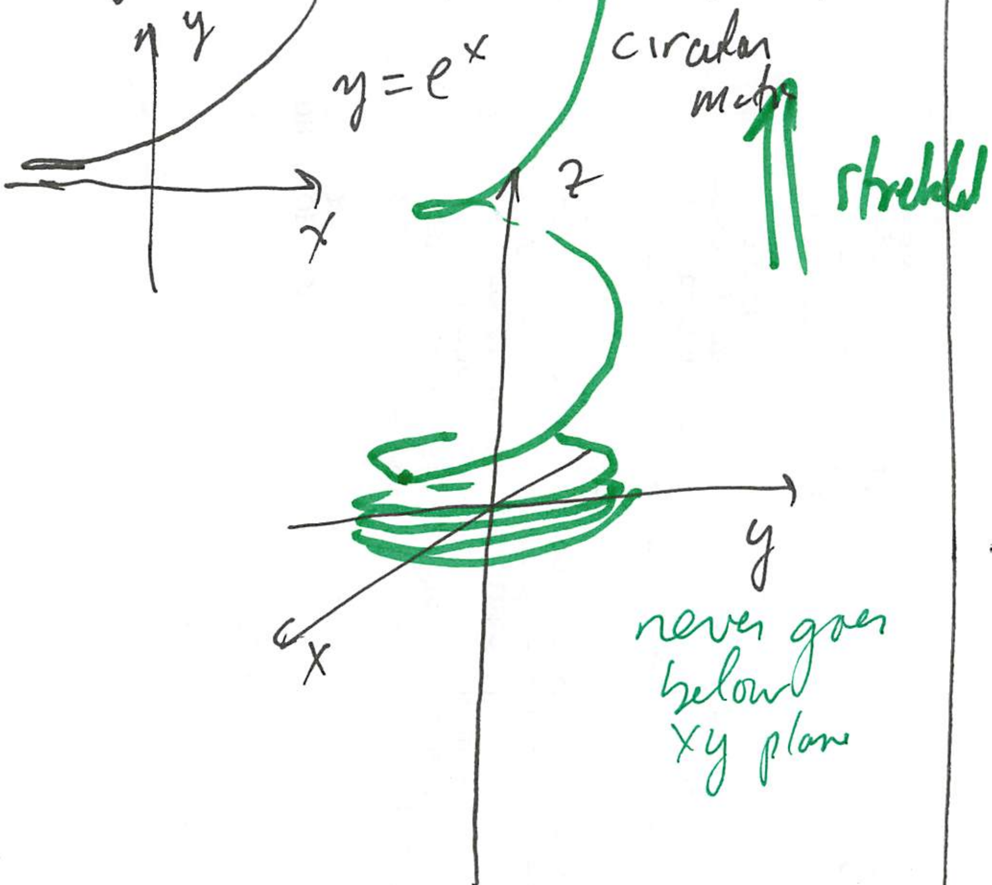
e.g. $\vec{r}(t) = \langle \underbrace{\cos t, \sin t}_{\text{circular}}, t^3 \rangle$

$= \langle \cos t, \sin t, 0 \rangle$

$+ \langle 0, 0, t^3 \rangle$



e.g. $\vec{r} = \langle \underbrace{\cos t, \sin t}_{\text{circular motion}}, e^t \rangle$



Calculus with VVFs. ①

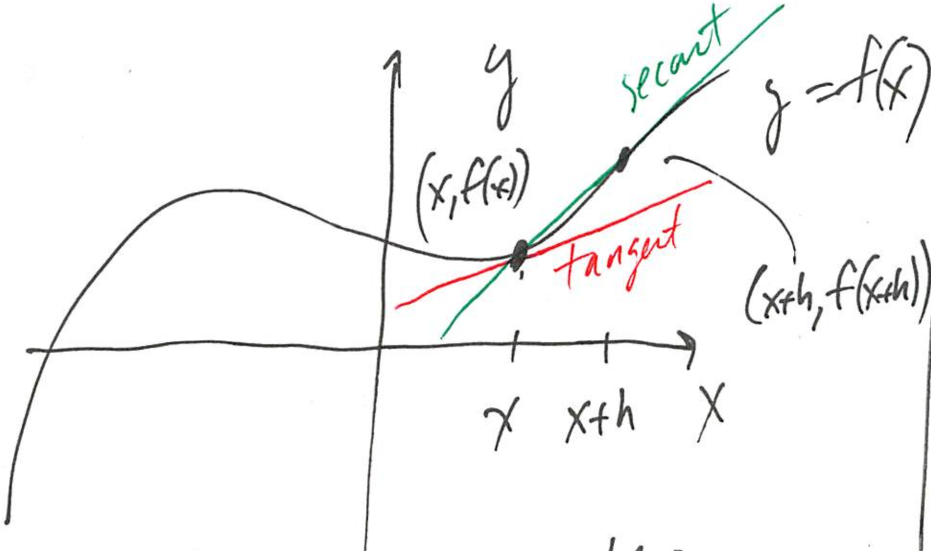
derivatives / integrals.

Recall M131: def of derivation.

$y = f(x)$

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

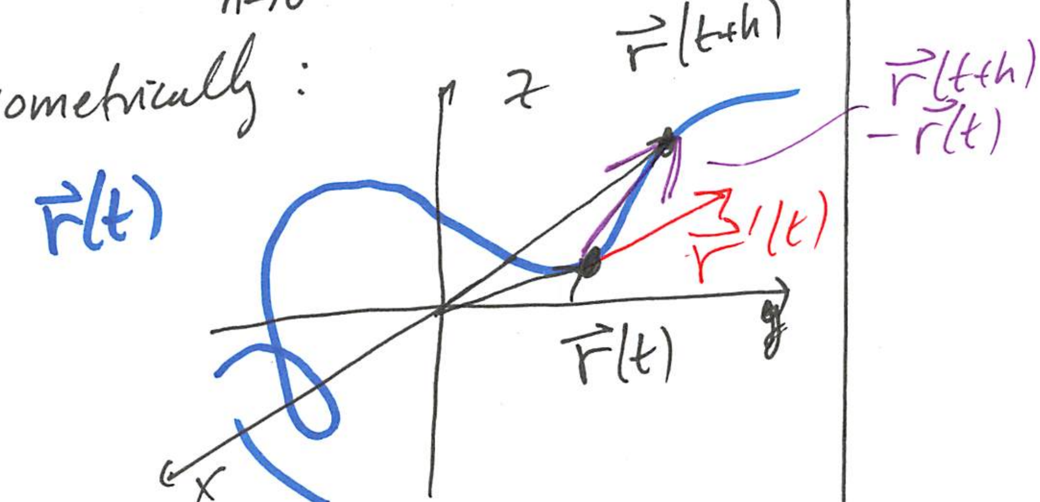
Idea: getting the slope of tangent line to graph of $y = f(x)$ by taking the limit of slopes of secant lines.



Want to mimic this for VVFs.

$$\vec{r}'(t) := \lim_{h\to 0} \frac{1}{h}\left(\vec{r}(t+h) - \vec{r}(t)\right)$$

Geometrically:



$\vec{r}\,'(t)$ = limiting vector and it's tangent to the graph @ $\vec{r}(t)$.

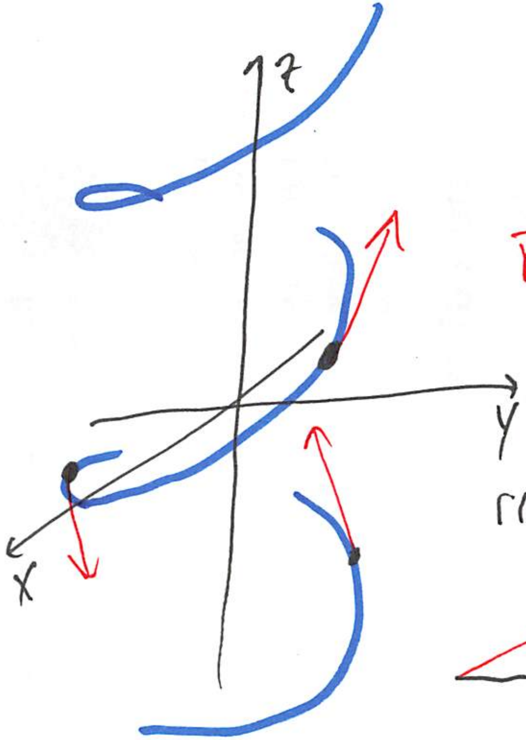
To actually compute $\vec{r}\,'(t)$, just do usual differentiation on the component functions.

e.g. $\vec{r}(t) = \langle t^2, t^3, \sin t\rangle$

$\vec{r}\,'(t) = \langle 2t, 3t^2, \cos t\rangle$

e.g. $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle$



side view

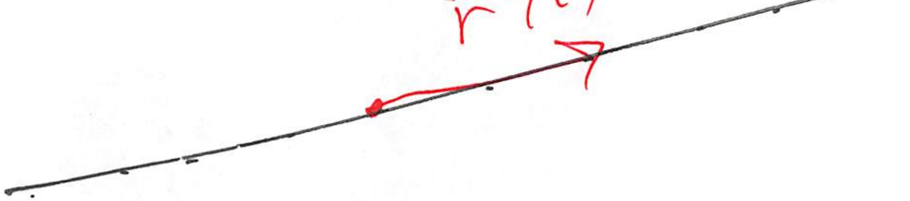
e.g. $\vec{r}(t) = \langle 2t, 3t+1, -t \rangle$

line

$\vec{r}'(t) = \langle 2, 3, -1 \rangle$

graph is line

$\vec{r}'(t) =$ direction vector



differentiation rules for VVFs

1. sum, difference rule.

$$\frac{d}{dt}\left(\vec{f}(t) \pm \vec{g}(t)\right) = \frac{d}{dt}\vec{f}(t) \pm \frac{d}{dt}\vec{g}(t)$$

2. scalar mult

$$\frac{d}{dt}\left(c(t)\vec{f}(t)\right) = c\vec{f}'(t) + c'\vec{f}(t)$$

3. dot product

$$\frac{d}{dt}\left(\vec{f} \cdot \vec{g}\right) = \vec{f}' \cdot \vec{g} + \vec{f} \cdot \vec{g}'$$

4. $$\frac{d}{dt}\left(\vec{f} \times \vec{g}\right) = \vec{f}' \times \vec{g} + \vec{f} \times \vec{g}'$$

cross product

Integration is done component-wise.

e.g. $\int_0^1 \vec{r}(t)\,dt$

$\vec{r}(t) = \langle t, t^2, t^3 \rangle$

means

$$\left\langle \int_0^1 t\,dt, \int_0^1 t^2 dt, \int_0^1 t^3 dt \right\rangle$$

$$= \left\langle \tfrac{1}{2}t^2 \Big|_0^1, \tfrac{1}{3}t^3 \Big|_0^1, \tfrac{1}{4}t^4 \Big|_0^1 \right\rangle$$

$$= \left\langle \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4} \right\rangle$$

Can also do indefinite integrals.
output is a VVF + constant vector that's not determine

e.g. $\int \langle t, t^2 \rangle dt$

$$= \left\langle \tfrac{1}{2}t^2, \tfrac{1}{3}t^3 \right\rangle + \vec{C}$$

const vector.

e.g. suppose $\vec{r}\,'(t) = \langle t^2, t, \sin t \rangle$
find $\vec{r}(t)$ given that $\vec{r}(0) = \vec{0}$

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt$$
$$= \left\langle \tfrac{1}{3}t^3, \tfrac{1}{2}t^2, -\cos t \right\rangle + \vec{C}$$

need $\vec{r}(0) = \vec{0}$.
plug in $t=0$, solve for $\vec{C}$.

$$\vec{r}(0) = \vec{C} + \langle 0,0,-1\rangle \overset{?}{=} \vec{0}$$

$$\Rightarrow \vec{C} = \langle 0,0,1\rangle$$

Ans: $\vec{r}(t) = \left\langle \tfrac{1}{3}t^3, \tfrac{1}{2}t^2, 1-\cos t \right\rangle$

$$\vec{f} = \langle t, t^2, t^3\rangle$$
$$\vec{g} = \langle 0, t, t^2\rangle$$
$$\vec{f}\times\vec{g} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & t^2 & t^3 \\ 0 & t & t^2 \end{pmatrix}$$
$$= \ldots \langle 0, -t^3, t^2\rangle$$
$$\frac{d}{dt}(\vec{f}\times\vec{g}) = \langle 0, -3t^2, 2t\rangle$$

$$\vec{f}\,'\times\vec{g} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & t & t^2 \end{pmatrix}$$
$$+\ \vec{f}\times\vec{g}\,' = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & t^2 & t^3 \\ 0 & 1 & 2t \end{pmatrix}$$

$\boxed{\text{Example of cross product rule.}}$