Exam 1 next week. Topics TBA.

TA will run a review session TBA

ch 14. Functions of several variables

usual calc. $y = f(x)$ — 1 input, 1 output

ch 13: VVFs $\vec{r}(t)$ — 1 input, 3 scalar outputs

Now $z = f(x,y)$ — 2 inputs, 1 output

$w = g(x,y,z)$ — 3 inputs, 1 output

could have more indep. vars.

$g(x_1, \dots, x_n)$ — usually take $n = 2, 3$

we can consider domains, ranges

e.g. $f(x,y) = x^2 + y^2$.

domain: set of allowed inputs. all x, y.

range: set of outputs. all real numbers ≥ 0.

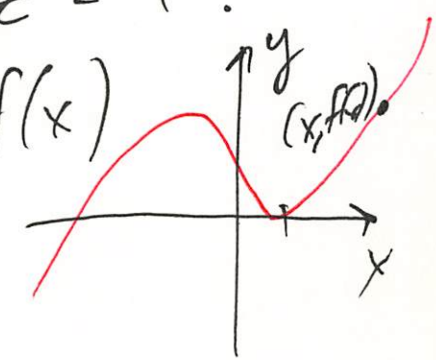
e.g. $f(x,y) = \sin(xy)$

domain: all x, y.

range: $z = \sin(xy)$

$-1 \leq z \leq 1$.

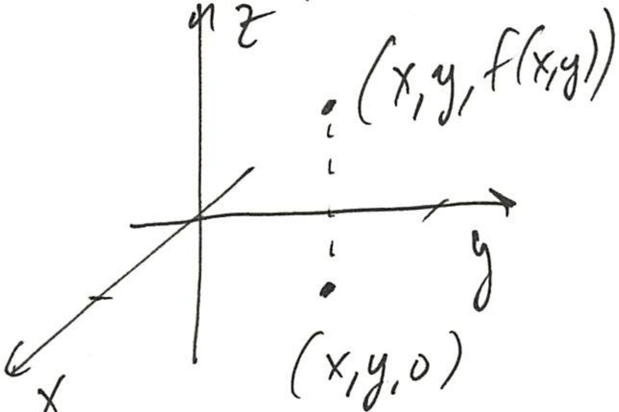
graphing. $y = f(x)$



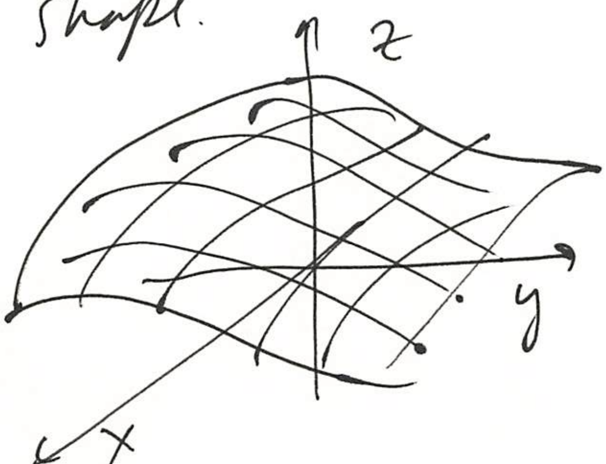
If have $z = f(x,y)$, can plot in 3D.

x, y : inputs.

z : value of $f(x,y)$

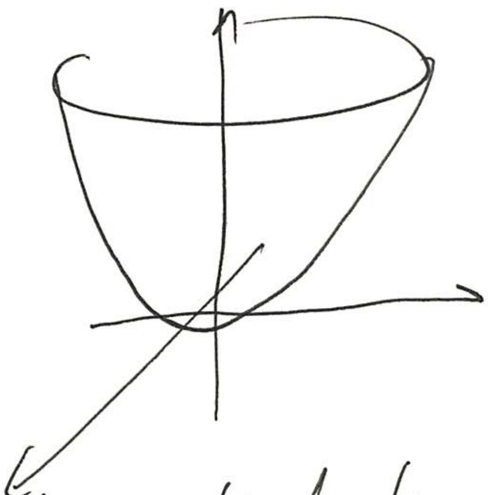


typically get a 2D shape.



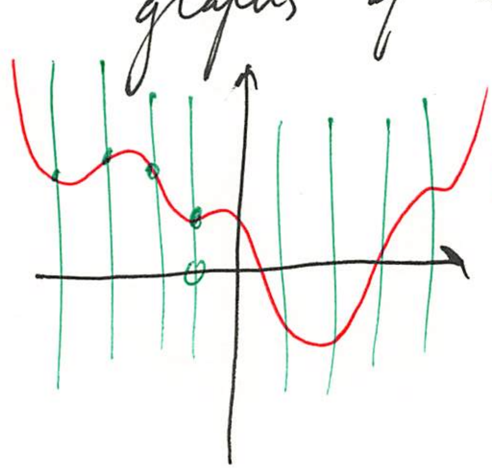
graph is called a __surface__ in 3D.

e.g. $z = x^2 + y^2$

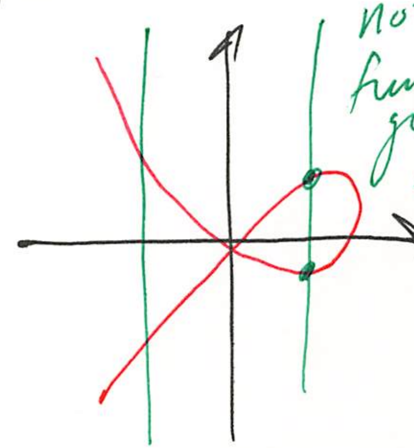


graph is paraboloid.

recall vertical line test for graphs of a function.

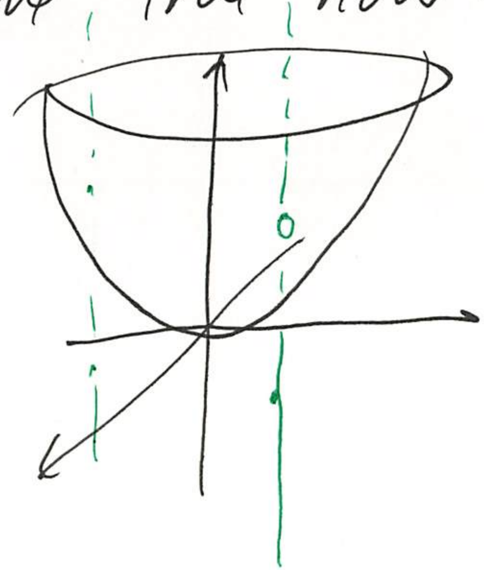


is



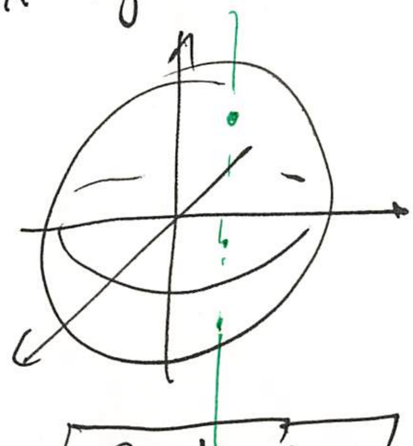
not function graph

Same true now.



vertical lines meet shape in at most 1 pt.

e.g. $x^2+y^2+z^2=1$ sphere



not graph of a function.

$z=\sqrt{\cancel{}\,1-x^2-y^2}$ upper hemisphere

$z=-\sqrt{1-x^2-y^2}$ lower.

Similarly, ellipsoids not graphs of functions.
hyperboloids not graphs of fn.
hyperbolic paraboloid is

$$z=x^2-y^2$$

e.g. what about $w=g(x,y,z)$?
Can we graph this?
we need 4 independent coordinates.

3 for x, y, z

1 for w

strictly speaking no.
But there are other methods to attack.

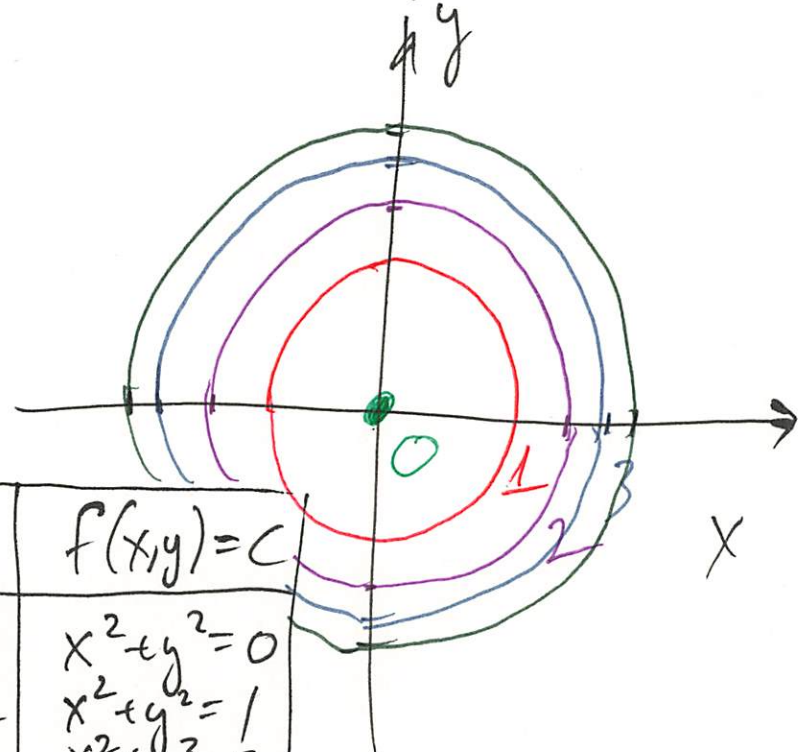
Level curves / contour graph

method to get geometric picture of a graph without drawing the 3D graph. (for functions of 2 vars)

Idea: set $f(x,y) =$ constant C for various choices of C. draw these graphs in the x,y plane. get a "map" of the shape in 3D.

e.g. paraboloid
$$z = x^2+y^2 = f(x,y)$$

choose different C and draw $f(x,y) = C$.

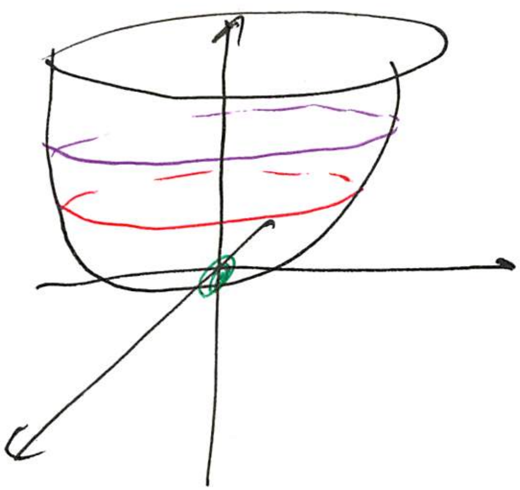


C	$f(x,y)=C$
0	$x^2+y^2=0$
1	$x^2+y^2=1$
2	$x^2+y^2=2$
3	$x^2+y^2=3$
4	$x^2+y^2=4$

These ~~pro~~ represent paths of constant height on the 3D graph.

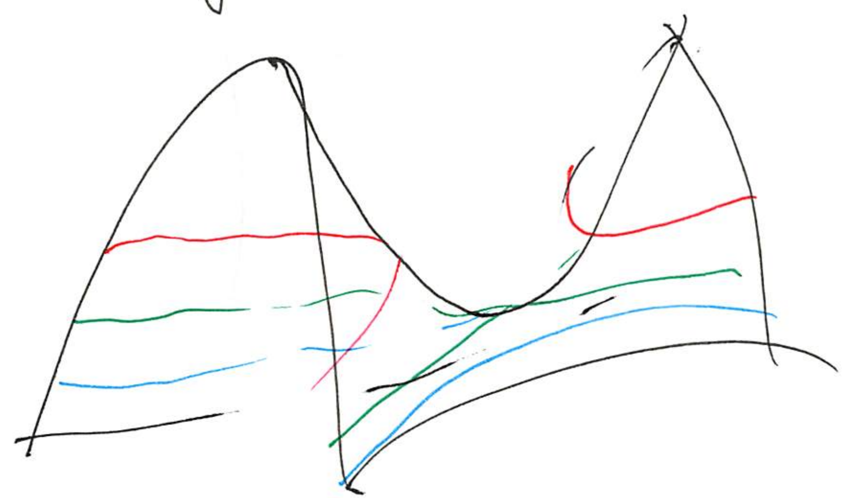
curves close $\Longleftrightarrow$ ~~very~~ ~~steep~~ steep.

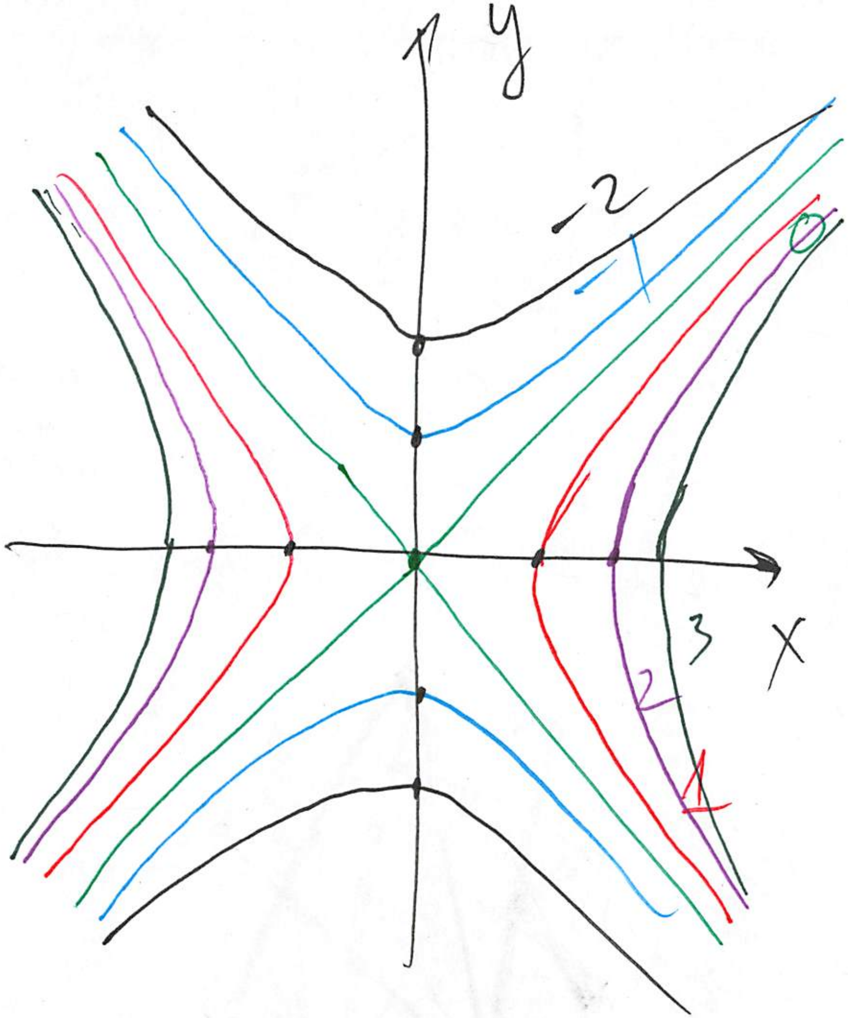
curves spread out $\Longleftrightarrow$ flat.



e.g. $z = x^2 - y^2$ hyperbolic paraboloid.

C	$x^2-y^2=C$
0	$x^2-y^2=0 \Leftrightarrow (x+y)(x-y)=0$
1	$x^2-y^2=1$ hyperbola.
2	$x^2-y^2=2$
3	$x^2-y^2=3$
-1	$x^2-y^2=-1$
-2	$x^2-y^2=-2$





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e.g.

$g(x,y,z) = x^2+y^2+z^2$

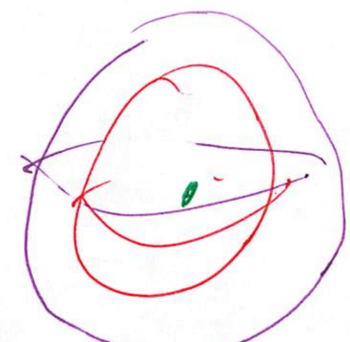
can graph $x^2+y^2+z^2 = C$

for different C.

get "level surfaces"

$x^2+y^2+z^2 = C$

$\longleftrightarrow$ sphere @ origin radius is $\sqrt{C}$

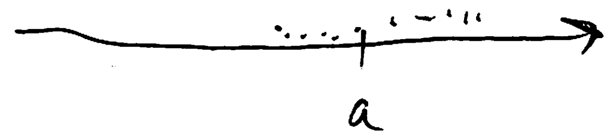


14.2 Limits and continuity

Recall def of continuous function from M131:

For $y = f(x)$, we say $f(x)$ is continuous @ $x = a$ if

1. $f(a)$ is defined..
2. $\lim_{x \to a} f(x)$ exists
3. limiting value in 2 equals $f(a)$



eg $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$

claim f is continuous at $x = 2$.

1. $f(1) = 2$ ✓
2. $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{x^2-1}{x-1}$
   $= \lim_{x \to 1} \dfrac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$
   $= \lim_{x \to 1} x+1 = 2$ ✓
3. $f(1) = \text{limit} = 2$ ✓

now $f(x,y)$ is continuous at (a,b) if

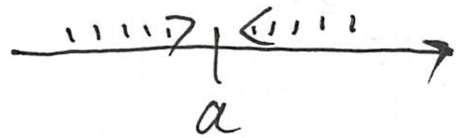
1. $f(a,b)$ defined
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exist.
3. limiting value in 2 agrees with 1

2 is now very different.

$\lim\limits_{(x,y)\to(a,b)} f(x,y)$ means

we must consider <u>all possible ways</u> to approach (a,b) in the xy plane

only 2 ways to approach a.



Calc 1

∞ many ways to approach

calc 3

For the limit to exist, it must be independent of any of these approaches

Typical problem: show that limits don't exist.

e.g. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

domain is all $(x,y) \neq (0,0)$ because denom $= 0$ at $(0,0)$

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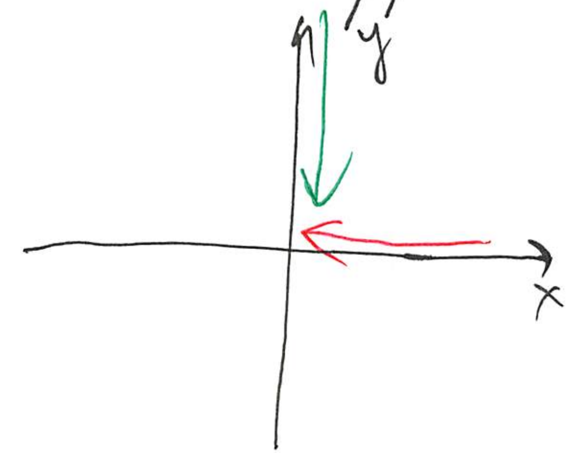
$f(x,y)$ is continuous everywhere except possibly the origin.

Can we define $f(0,0)$ to make it continuous?

need to consider

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

Consider 2 approaches:



red: put $y=0$
consider

$$\lim_{x\to 0} f(x,0)$$

$$= \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

green: put $x=0$
consider

$$\neq$$

$$\lim_{y\to 0} f(0,y)$$

$$= \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

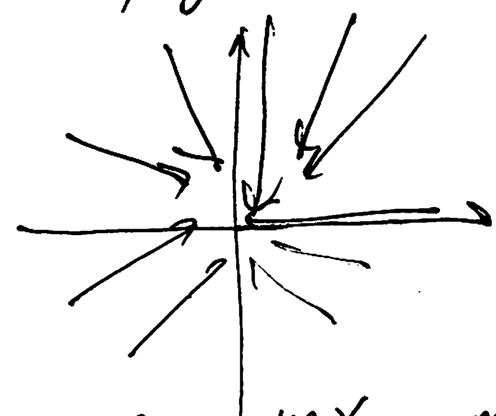
$\Longrightarrow$ limit D.N.E.

e.g. $\lim_{(x,y)\to(0,0)} f(x,y)$

$$f(x,y) = xy^2/x^2+y^4$$

– try $y=0$, $x\to 0 \Longrightarrow 0$

– try $x=0$, $y\to 0 \Longrightarrow 0$



– try $y = mx$, m fixed.
get same answer!

– try $x=y^2$, get different answer

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$$y^4 / y^4 + y^4 = \frac{1}{2} \frac{y^4}{y^4}$$

$$\to \frac{1}{2} \neq 0$$

$$\left(\lim_{y \to 0} f(y^2, y) = \frac{1}{2} \right)$$