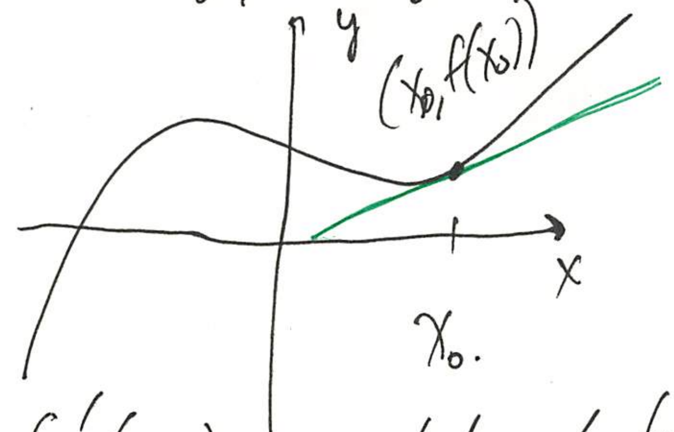
Exam will cover through 13.4
Review session with TA.

ch 14.
Last time: $z = f(x,y)$
$w = g(x,y,z)$
graphing
contour graphs / level curves.
limits / continuity.

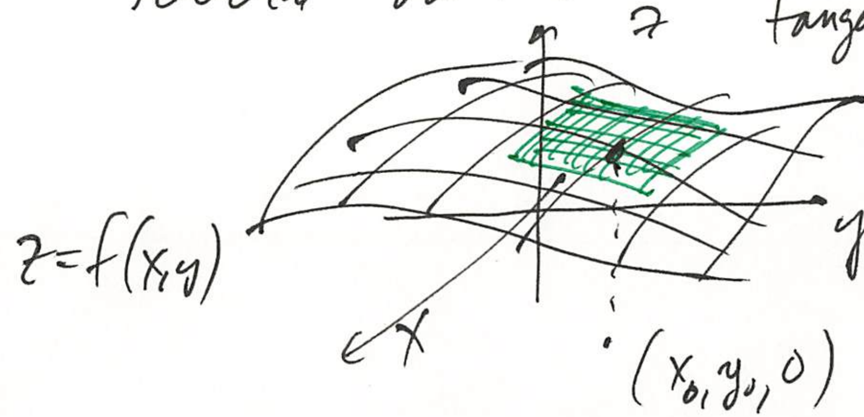
Today: partial derivatives
tangent planes / linear approximation

① usual calc: $y = f(x)$
$\frac{dy}{dx} = y' = f'(x)$ derivative



$f'(x_0)$ = slope of tangent line.

want similar for fns of several variables
should have tangent plane not a line.



now have 2 independent vars x, y
get 2 different derivatives, one for x, one for y.

"partial derivatives" of $f(x,y)$

Idea: fix all variables except for one, and see how the function changes as we vary this variable.

Def. $f(x,y)$ (x_0, y_0) pt in domain of f.
Then the partial derivative of f w.r.t. x at (x_0, y_0) is

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

Similarly,

②

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

$\longrightarrow$ 2 derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}.$$

e.g. $w = g(x,y,z)$

$$\Longrightarrow \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}$$

To compute, treat all fixed variables as constants, and differentiate as usual!

e.g. $f(x,y) = x^2 + y^2$

$$\frac{\partial f}{\partial x} = ?$$

diff. w.r.t x, pretend y is a constant.

$$\frac{\partial}{\partial x}(x^2+y^2) = 2x + 0$$
$$= 2x$$
$$\frac{\partial}{\partial y}(x^2+y^2) = 2y$$

e.g. $f(x,y) = x^3y + x^2y^2 + e^{xy}$

$$\frac{\partial f}{\partial x} = 3x^2y + 2xy^2 + e^{xy}\cdot y$$
$$\frac{\partial f}{\partial y} = x^3 + 2x^2y + e^{xy}\cdot x$$

e.g. $g(x,y,z) = x^2\sin y + e^{xyz}$

$$\frac{\partial g}{\partial x} = 2x\sin y + e^{xyz}yz$$
$$\frac{\partial g}{\partial y} = x^2\cos y + e^{xyz}xz$$
$$\frac{\partial g}{\partial z} = e^{xyz}xy$$

other notations:

$$z = f(x,y)$$

$\frac{\partial f}{\partial x}$, $\frac{\partial z}{\partial x}$, f_x, z_x

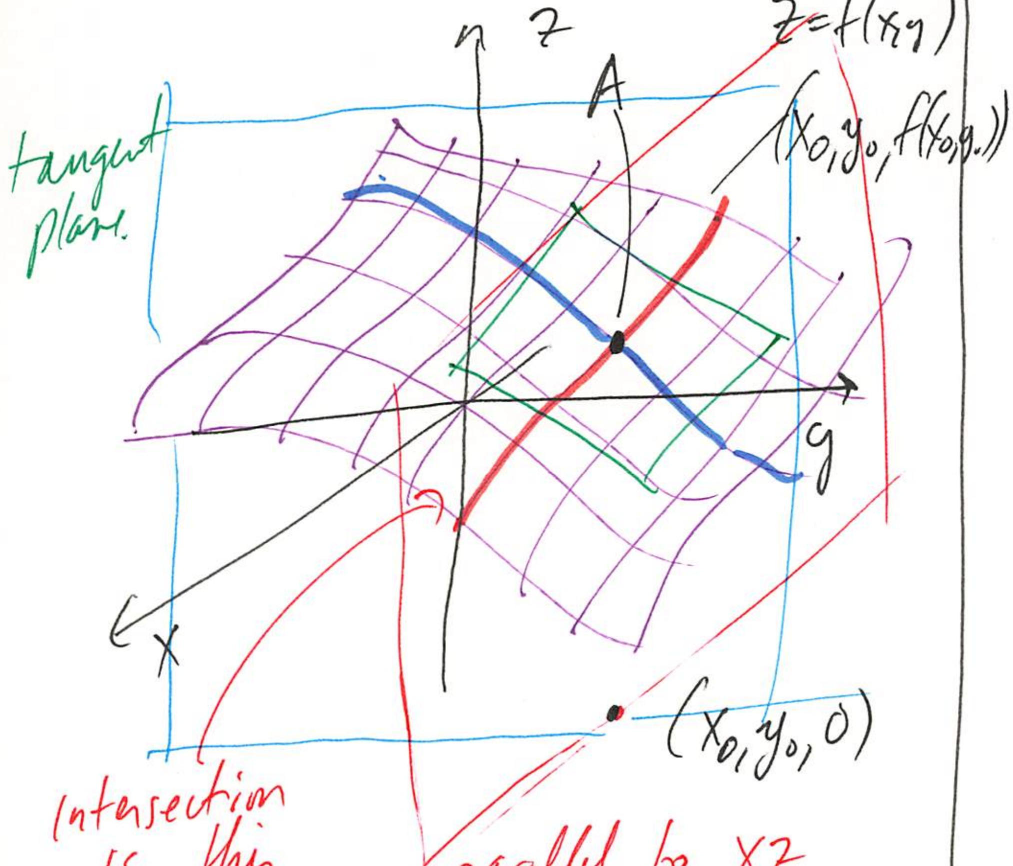
all mean same partial der.

sim. $\frac{\partial f}{\partial y}$, $\frac{\partial z}{\partial y}$, f_y, z_y.

<u>Geometric meaning</u>.

partial derivatives are related to slopes.

slopes of certain lines on the tangent plane.



intersection is this curve.

parallel to XZ coord plane

slope at A of this curve is $f_x(x_0, y_0)$

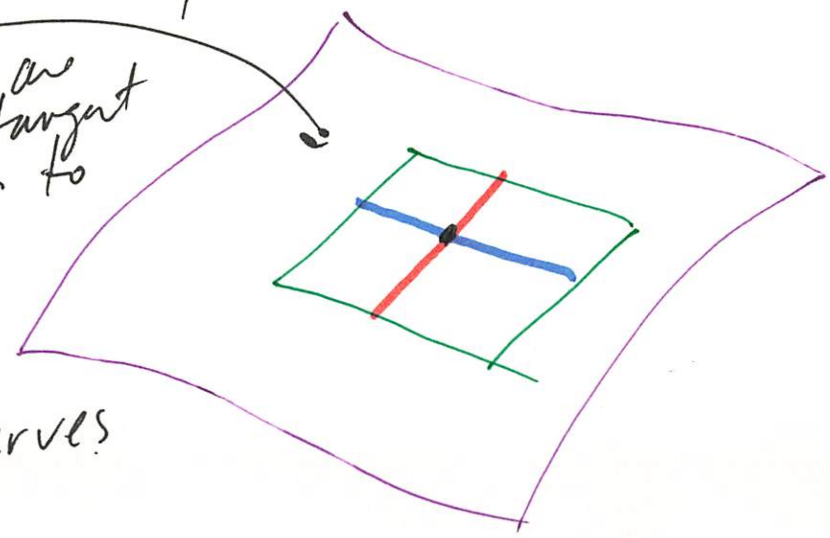
its tangent line lies in green plane.

If consider $f_y(x_0, y_0)$, get similar picture, with the red plane replaced by one parallel to the y-z plane.

Blue plane

$f_y(x_0, y_0)$ is slope of blue curve at that point.

there are the tangent lines to the red, blue curves



Have 2nd partial derivatives

$f \rightsquigarrow f_x, f_y$

since each can be diff. in 2 different ways, we get 4 possible 2nd partial derivatives.

$$\left.\begin{aligned} (f_x)_x &= f_{xx} \\ (f_x)_y &= f_{xy} \\ (f_y)_x &= f_{yx} \\ (f_y)_y &= f_{yy} \end{aligned}\right\} \text{4 things.}$$

e.g. $f = x^2y + y^3 + x^4$

$$f_x = 2xy + 4x^3$$

$$f_y = x^2 + 3y^2$$

$$f_{xx} = 2y + 12x^2$$

$$f_{xy} = 2x$$

$$f_{yx} = 2x$$

$$f_{yy} = 6y$$

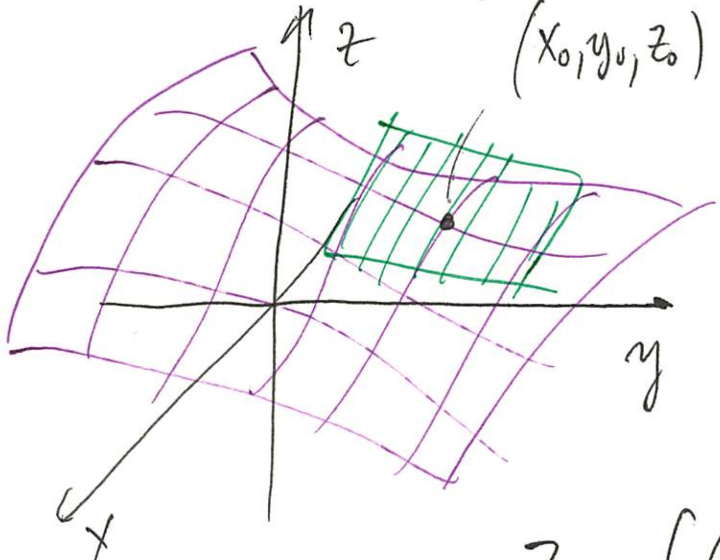
f_{xy} and f_{yx}: $=$; f_{xx}, f_{yy}: $\neq$

In fact, we always equality of mixed partials

$$f_{xy} = f_{yx}$$

Later: geometry of 2nd partial derivatives.

Tangent planes / linear approximation.



$z_0 = f(x_0, y_0)$

$$\boxed{z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)}$$

graph of a function itself namely

$z_0 + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

notice it has the same partial derivatives as f does at the point (x_0, y_0)

$\Longrightarrow$ it's the tangent plane.

- passes through $(x_0, \cancel{y}_0, z_0)$
- has same partial derivatives, so it contains the tangent lines to the red, blue curves.

e.g. $f(x,y) = x^2 y + y^3$

pt $(1, 1, 2) = (x_0, y_0, z_0)$

what's tangent plane?

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$f_x = 2xy$, $f_y = x^2 + 3y^2$

@ pt: $f_x = 2$

$f_y = 4$

so plane is

$$z - 2 = 2(x-1) + 4(y-1)$$

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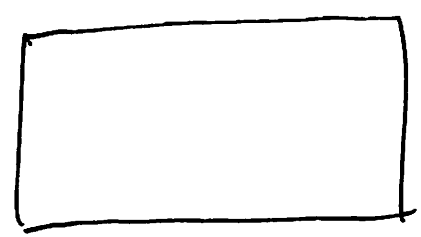
linear approximation.

using the tangent plane to approximate the graph of the function.

application: propagation of error in computations

e.g.

5 ± 0.1 cm



10 ± 0.1 cm

rectangle

Q: area, with reasonable err. est.

$z = f(x,y)$ $\quad x \to x + \Delta x$

$\qquad\qquad\quad y \to y + \Delta y.$

what is approximate change in z? ie

$f(x+\Delta x, y+\Delta y)$

$= z + \Delta z$

we want a reasonable approximation to Δz

Lin approx:

$\Delta z \approx f_x(x,y)\Delta x + f_{\cancel{x}y}(x,y)\Delta y.$

comes directly from tangent plane

$$\underbrace{z - z_0}_{\approx \Delta z} = f_x(x_0,y_0)\overbrace{(x - x_0)}^{\Delta x} + f_y(x_0,y_0)\underbrace{(\cancel{x} - y_0)}_{\Delta y.}$$

e.g. $A = xy$

$\Delta A \approx A_x \Delta x + A_y \Delta y$

$A_x = y, \; A_y = x$

$\Longrightarrow \Delta A \approx y\Delta x + x\Delta y.$

ΔA is going to give approximate error.

$x = 10$, $\Delta x = 0.1$

$y = 5$, $\Delta y = 0.1$

$$\Delta A = 5 \cdot 0.1 + 10 \cdot 0.1$$
$$= 1.5$$

$$\boxed{A = 50 \pm 1.5\ cm^2}$$