Last time: fns of several vars
partial derivatives
tangent planes

Today: chain rule for fns of several vars.
gradient vector

Chain rule: differentiating compositions of functions

e.g. $f(x) = \sin(x^2)$

$f'(x) = \cos(x^2) \cdot 2x$

$f(x) = (g \circ h)(x)$
$= g(h(x))$

$h = x^2, \quad g = \sin x$

chain rule: if $f = g \circ h$, ①

then $f'(x) = g'(h(x))\, h'(x)$

or: $\dfrac{df}{dx} = \dfrac{dg}{dh} \cdot \dfrac{dh}{dx}$

$g(h) = \sin(h)$
$h(x) = x^2.$

or y is fn of u
u is fn of x

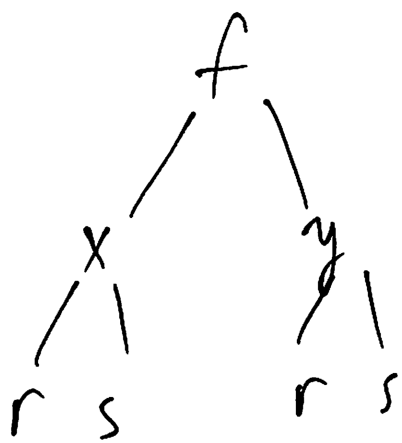
$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

with more than $\underline{1}$ indep. var:

1. have partial derivatives
2. have multiple contributions for a given derivative.

e.g. f depends on x, y
x depends on r, s
y depends on r, s

$\frac{\partial f}{\partial r} = ?$, $\frac{\partial f}{\partial s} = ?$

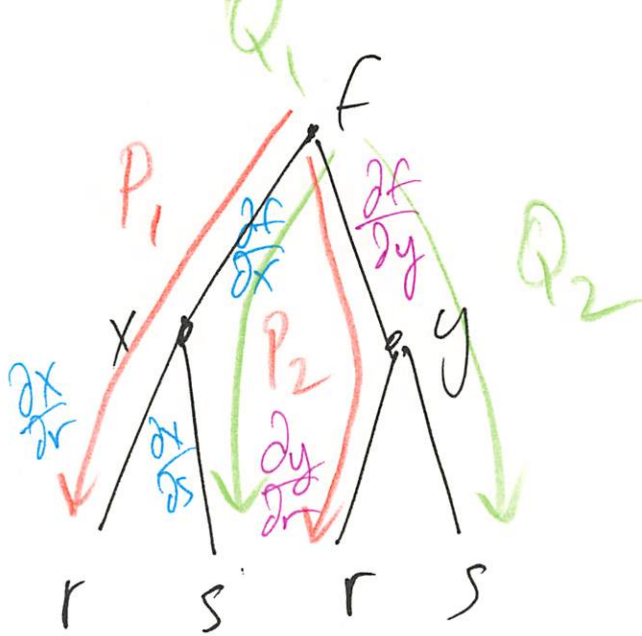


$(*)$

Suppose we want $\frac{\partial f}{\partial r}$.
we must take into account the dependence via x, y.

To compute $\frac{\partial f}{\partial r}$:

1. find all paths from f to r in $(*)$
2. put appropriate derivatives as labels of the path edges.
3. Take products of derivatives along paths, sum up the contributions from the different paths



(1) : P_1, P_2

(2) : edge labels.

(3) $P_1 : \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r}$, $P_2 : \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$

sum: $P_1 + P_2$

$$\boxed{\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}}$$

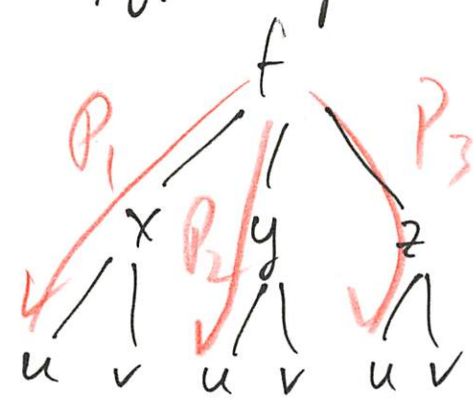
(3)

$\frac{\partial f}{\partial s} = ?$ comes from Q_1, Q_2.

$$\frac{\partial f}{\partial s} = \underbrace{\frac{\partial f}{\partial x} \frac{\partial x}{\partial s}}_{Q_1} + \underbrace{\frac{\partial f}{\partial y} \frac{\partial y}{\partial s}}_{Q_2}$$

---

e.g. f depends on x, y, z
x, y, z depend on u, v



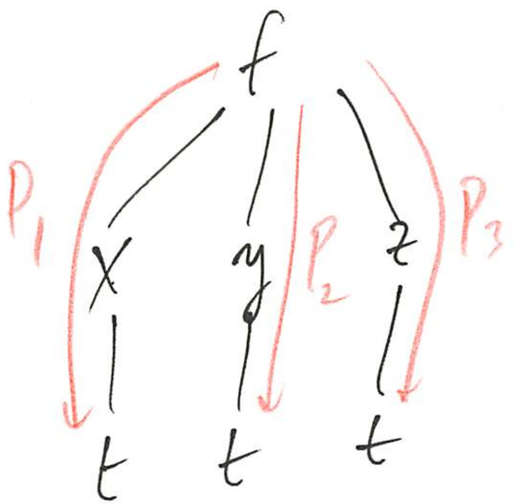
$$\frac{\partial f}{\partial u} = \underbrace{f_x x_u}_{P_1} + \underbrace{f_y y_u}_{P_2} + \underbrace{f_z z_u}_{P_3}$$

e.g. f depends on x, y, z

x, y, z depend on t

really only one indep var: t

$\frac{df}{dt}$ makes sense



$$\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial x}\frac{dx}{dt}}_{P_1} + \underbrace{\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}}_{P_2} + \underbrace{\frac{\partial f}{\partial z}\frac{dz}{dt}}_{P_3}$$

Directional derivatives / gradient.

Gradient: one of the most important constructions in the course.

suppose have $f(x, y)$.

we have $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

2 different measures of how f changes.

Gradient: collect these partials into a single vector.

$$\nabla f := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

gradient of f

∇f is pronounced

1. grad f
2. nabla f
3. atled f

} options

also just write "grad f" sometimes

e.g. $f(x,y) = x^2 + y^2$

$$\nabla f = \langle f_x, f_y \rangle$$
$$= \langle 2x, 2y \rangle$$

get different vectors for different inputs

e.g. $\nabla f(1,2) = \langle 2, 4 \rangle$

5

eg $f = \sin(x^2) + ze^y + z^3$.

3 vars.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$
$$= \langle 2x\cos(x^2), ze^y, e^y + 3z^2 \rangle$$

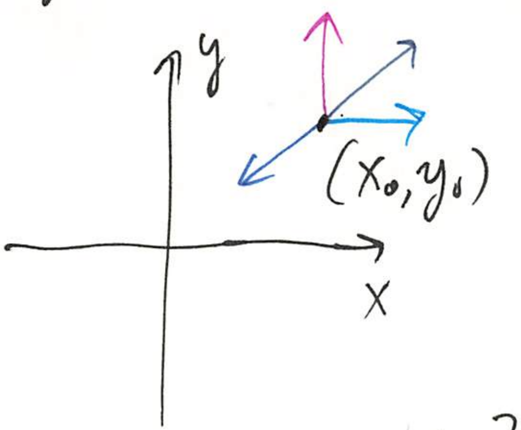
2 applications:

(a) directional derivative

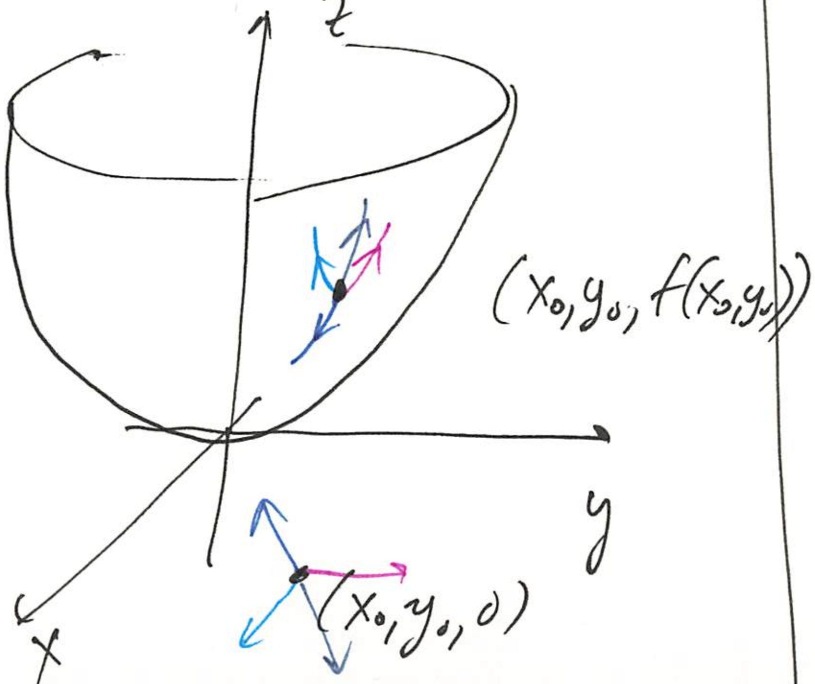
(b) relationship between ∇f and level curves of $f(x,y)$.

(a) idea: f changes by different amount depending on how we move.

$f(x,y) = x^2 + y^2$



different directions give different change in f



The directional derivative computes how f changes as we move in different directions

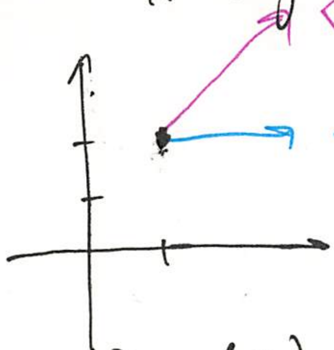
__Def__ Let f be a function of x, y. Let $\hat{u}$ be a unit vector. Then the directional derivative of f at (x_0, y_0) in the direction of $\hat{u}$ is

$$(D_{\hat{u}} f)(x_0, y_0) = (\nabla f) \cdot (\hat{u}) \Big|_{(x_0, y_0)}$$

e.g.

$f = x^2 + y^2$ $\quad (x_0, y_0) = (1,2)$

$\langle 1,1 \rangle = \vec{u}_2$



$\langle 1,0 \rangle = \vec{u}_1$ $\quad |\vec{u}_1| = 1$

$|\vec{u}_2| = \sqrt{2}$

$D_{\hat{u}_1}(f) = ?$

$\nabla f = \langle 2x, 2y \rangle$

@ $(1,2)$ get $\langle 2,4 \rangle$

$\hat{u}_1 = \vec{u}_1$

$D_{\hat{u}_1}(f) = \langle 2,4 \rangle \cdot \langle 1,0 \rangle$

$= 2$

$D_{\hat{u}_2}(f) = ?$

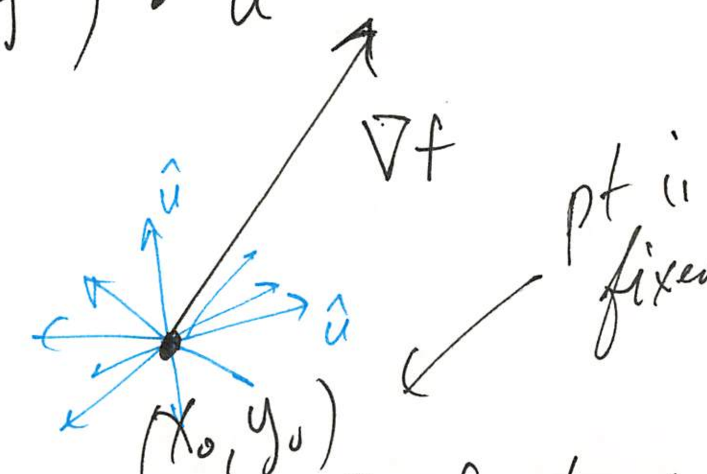
$\hat{u}_2 = \vec{u}_2 / \sqrt{2} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$D_{\hat{u}_2}(f) = \langle 2,4 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$= \frac{6}{\sqrt{2}} > 2$

(b) connection with level curves.

$(\nabla f) \cdot \hat{u}$

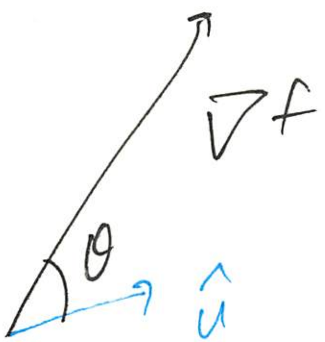


Q: how does $D_{\hat{u}} f$ change as we vary $\hat{u}$?

Fact: we get numbers in some bounded interval

$$\min \le D_{\hat{u}}(f) \le \max$$

$$\min = -\max$$



$$\nabla f \cdot \hat{u} = |\nabla f| \cos\theta$$

max occurs when $\cos\theta = 1 \iff \theta = 0$

$\implies$ max directional der. occurs when $\hat{u}$ same direction as the gradient.

$\implies$ gradient points in the direction of steepest ascent.

also, min occurs when $\cos\theta = -1 \iff \theta = \pi$

$\implies$ $-\nabla f$ points in direction of steepest descent.

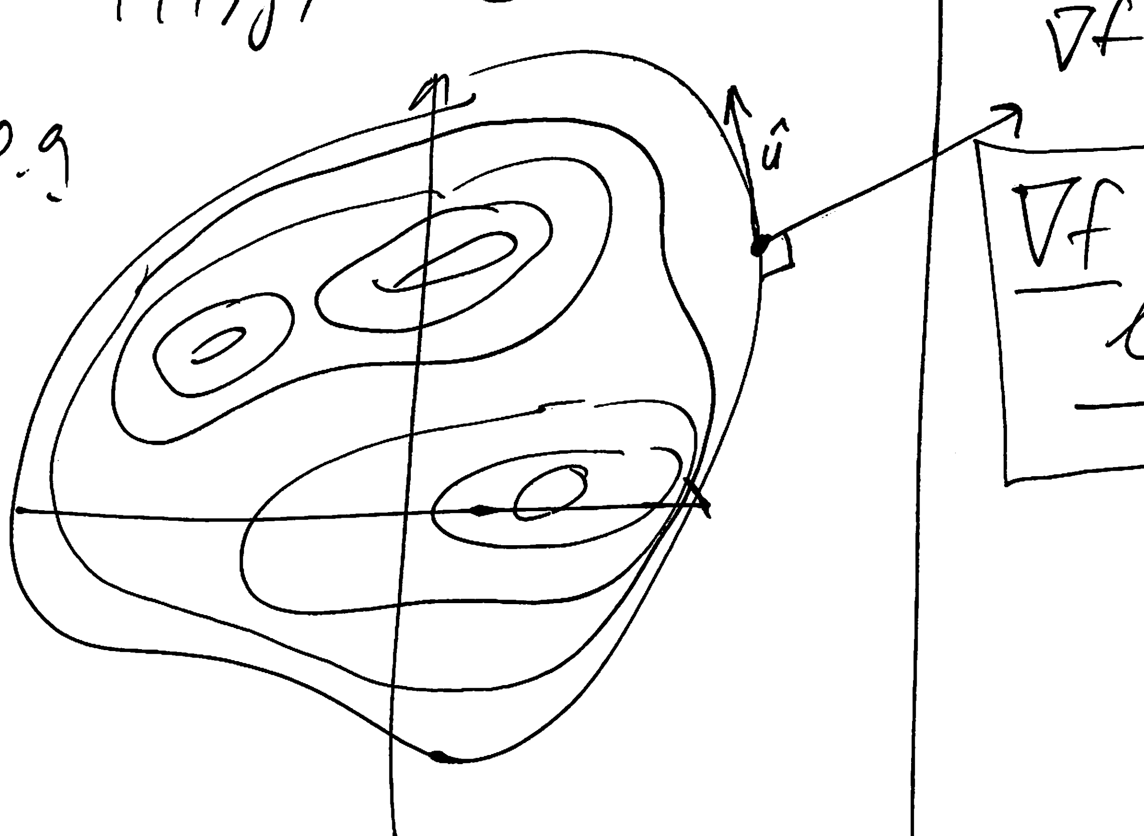
$D_{\hat{u}}(f) = 0 \iff \cos\theta = 0$

$\iff \theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

$\implies$ $\perp$ to gradient, the function value doesn't change.

recall level curves: graphs in x,y plane of form
$$f(x,y) = C \quad (\text{const})$$

e.g.



$$\boxed{\nabla f \perp \text{to level curves}}$$