①

max/min problems

$f(x,y)$ want to max/min.

① find critical points
$$\nabla f = \vec{0}$$

② test these pts using Hessian.

$$D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

① $D>0$ ⓐ $f_{xx}<0$ MAX
ⓑ $f_{xx}>0$ MIN

② $D<0$ SADDLE PT

③ $D=0$ TEST FAILS.

e.g. $f = x^4+y^4-4xy+1$

Find and classify crit pts.

$$\nabla f = \langle f_x, f_y \rangle \overset{?}{=} \vec{0}.$$

① $f_x = 4x^3 - 4y = 0$

② $f_y = 4y^3 - 4x = 0$

① $\rightarrow$ $y = x^3$ (*) use this in ②

$\Rightarrow$ $x^9 - x = 0$

$x(x^8-1) = 0$

$x(x^4+1)(x^4-1) = 0$

$x(x^4+1)(x^2+1)(x^2-1) = 0$

$\underline{x}(x^4+1)(x^2+1)\underline{(x+1)}\underline{(x-1)} = 0$

$x = 0, -1, 1$

use in (*) to find y.

$(0,0)$, $(1,1)$, $(-1,-1)$
crit. pts.

$f_{xx} = 12x^2$, $12y^2 = f_{yy}$

$f_{xy} = f_{yx} = -4$

$$D = \det\begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$= 144x^2y^2 - 16$$

$(0,0)$: $D < 0 \Rightarrow$ saddle pt

$(1,1)$: $D > 0$, $f_{xx} > 0 \Rightarrow$ MIN

$(-1,-1)$: $D > 0$, $f_{xx} > 0 \Rightarrow$ MIN.

<u>Rem</u> "Test Fails" means we really can't tell what's happening.

e.g. $f_1 = x^4 + y^4$, $f_2 = -x^4 - y^4$, $f_3 = x^4 - y^4$ — $(0,0)$ crit pt, $D = 0$ @ $(0,0)$ for all 3.

f_1: MIN, f_2: MAX, f_3: SADDLE

Lagrange multipliers

Max/Min problems where the domain is subject to a constraint.

$f(x,y)$ fn to max/min.

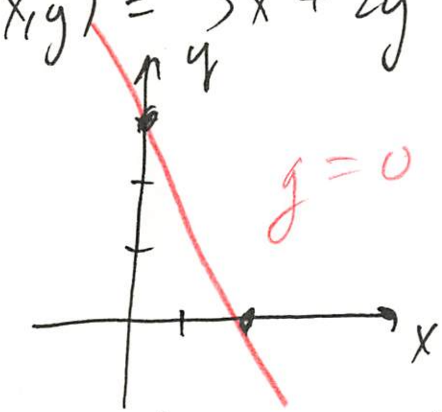
$g(x,y) = 0$ graph constraining the value of x, y we consider.

e.g. find distance between the origin and the line

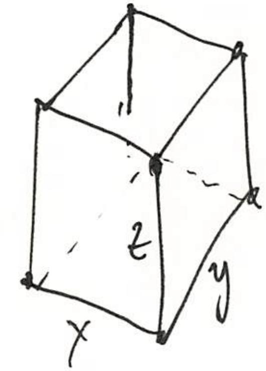
$$3x + 2y = 6$$

$$f(x,y) = x^2 + y^2$$

$$g(x,y) = 3x + 2y - 6 = 0$$



e.g. Find max volume of open-topped box that can be made from $12m^2$ of material.



$$f(x,y,z) = xyz$$

$$g(x,y,z) = xy + 2xz + 2yz - 12 = 0$$

Idea: max/min should occur when ∇f is a multiple of ∇g. Introduce a new variable λ and consider the system of equations

$$\left.\begin{array}{l} \nabla f = \lambda \nabla g \\ g = 0 \end{array}\right\}.$$

Same # of equations as variables!

e.g. $f = x^2 + y^2$

$g = 3x + 2y - 6$

$\left.\begin{array}{l} \nabla f = \lambda \nabla g \\ g = 0 \end{array}\right\}$

$\nabla f = \langle 2x, 2y \rangle, \ \lambda \nabla g = \langle 3\lambda, 2\lambda \rangle$

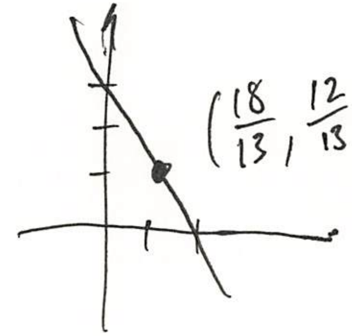
$\Longrightarrow \left.\begin{array}{l} 2x = 3\lambda \\ 2y = 2\lambda \\ 3x + 2y = 6 \end{array}\right\}$ 3 eqns, 3 vars

$y = \lambda, \ x = \frac{3}{2}\lambda$

$\Rightarrow \frac{9}{2}\lambda + \frac{4}{2}\lambda = 6$

$\lambda = 12/13$

$\Rightarrow (x, y) = \left(\frac{18}{13}, \frac{12}{13}\right)$



← unique pt of interest, geom. must be a min.

e.g. box problem

$f = xyz$

$g = xy + 2xz + 2yz - 12 = 0$

$\nabla f = \lambda \nabla g \Longrightarrow$

$\partial/\partial x$ ① $xyz = \lambda(y + 2z)x$

$\partial/\partial y$ ② $yxz = \lambda(x + 2z)y$

$\partial/\partial z$ ③ $zxy = \lambda(2x + 2y)z$

④ $xy + 2xz + 2yz = 12$

$x, y, z \neq 0, \ \lambda \neq 0$ for max vol.

T

1 = 2 $\Rightarrow$

$$\lambda(xy+2xz) = \lambda(xy+2yz)$$

$\lambda \neq 0 \Rightarrow$ can divide it away

$$\cancel{xy}+2xz = \cancel{xy}+2yz$$

$$\Rightarrow \boxed{x=y}$$

2 = 3 $\Rightarrow$ $\boxed{y=2z}$

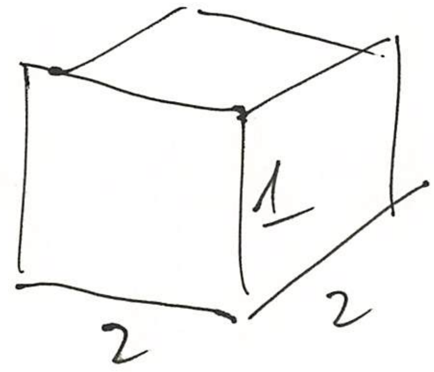
put into 4

$$4z^2+4z^2+4z^2=12$$

$$\Rightarrow z^2=1$$

$$\Rightarrow z=1$$

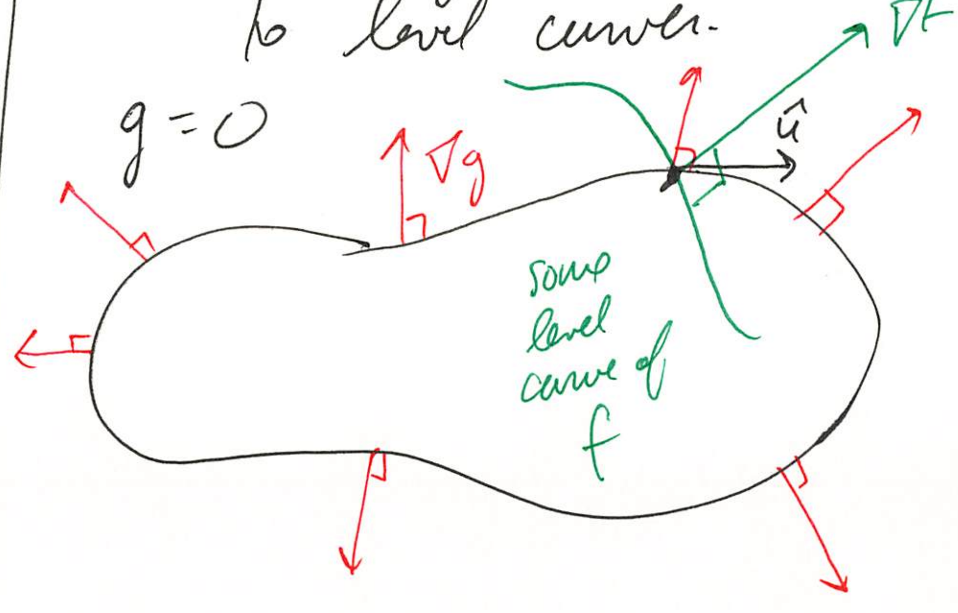
$$\Rightarrow y=x=2.$$



vol = 4.

Reason:

Recall gradients are $\perp$ to level curves.



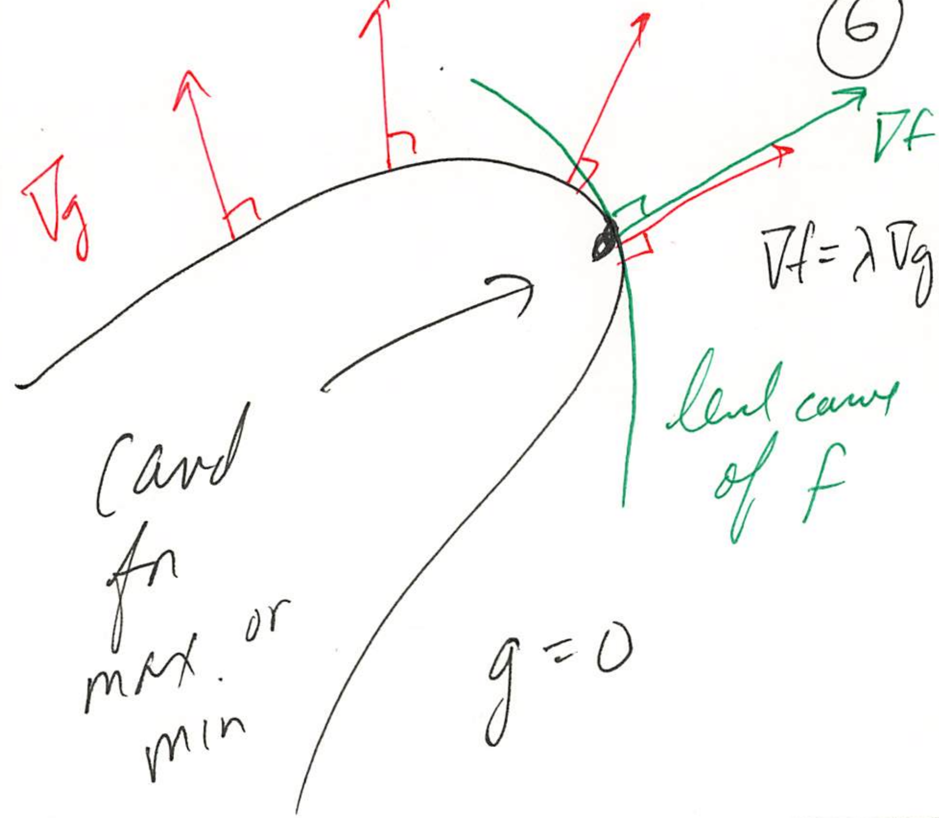
if $\hat{u}$ is tangent to $g=0$, then moving in that direction along g will change f by

$$D_{\hat{u}} f = \nabla f \cdot \hat{u}$$

move in this direction and stop when I can't increase f. This happens when

$\nabla f \cdot \hat{u} = 0$ or

∇f itself $\perp$ to graph of g.

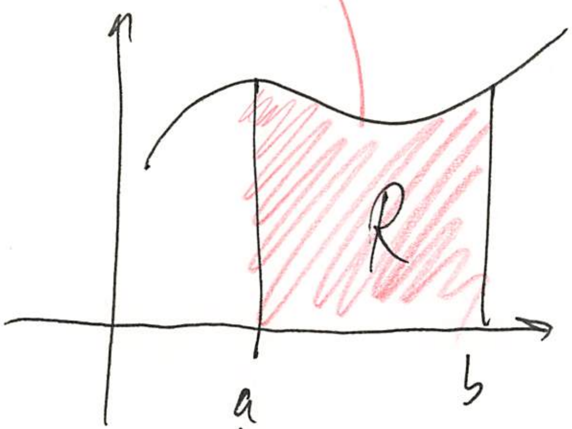


Ch 15 Integration.

multiple integrals (meaning we have more than one variable in the integral)

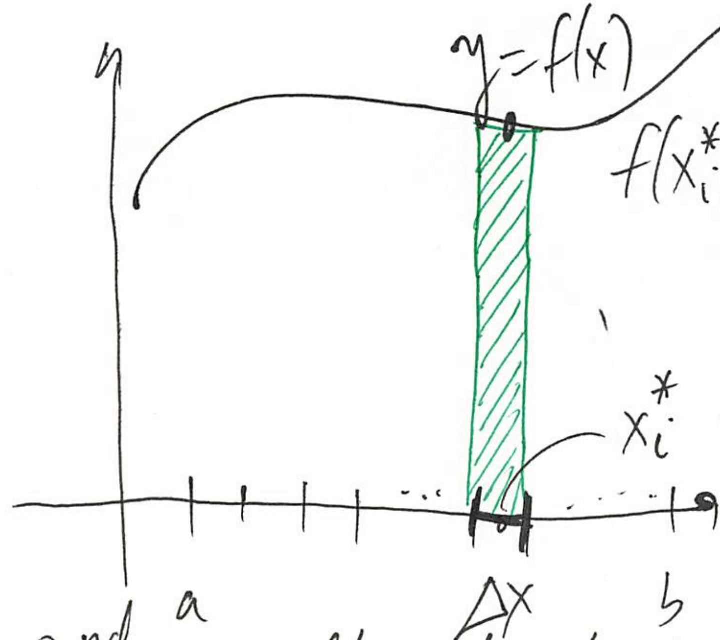
Recall def of integral in Calc 1/2

$\int_a^b f(x)\,dx$ = area under graph of $y=f(x)$ over $a \le x \le b$



To make precise: approximate R with rectangles, then take limit.

1st subdivide base $[a,b]$ into pieces of length Δx



2nd: build rectangle by choosing pt on graph over the base.

take sample pt x_i^*
get rectangle base Δx
height $f(x_i^*)$

3rd: add up these areas.

Area $\approx$ area of these rectangles

If we have N of them, we get

$$\text{Area} \approx \sum_{i=1}^{N} f(x_i^*) \Delta x$$

4th: take lim as $N \to \infty$.

If all goes well, the limit exists, is a number, and that number is what we denote

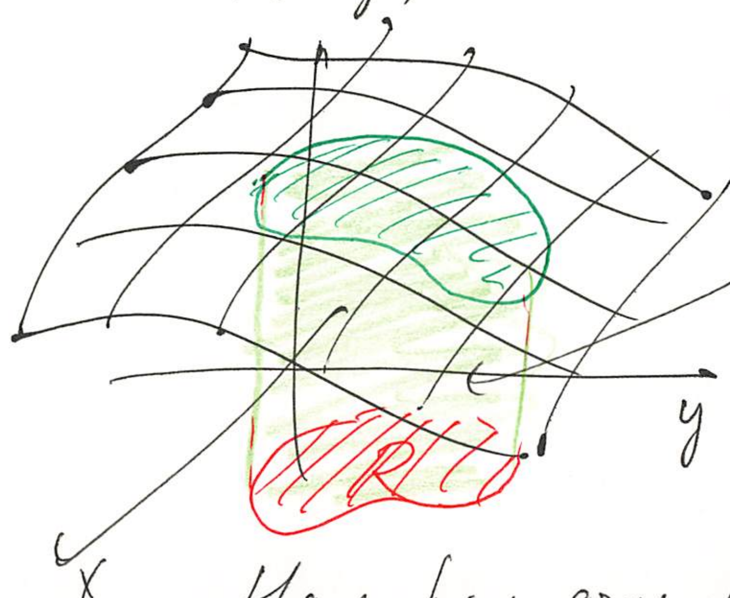
$$\int_a^b f(x)\,dx$$

Want similar construction for functions of more than 1 variable.

$f(x,y)$

area $\Leftrightarrow$ volume

$z = f(x,y)$



graph of $f(x,y)$

want volume

Have have some base region R in xy plane

We will introduce a definite integral computing the volume under a graph and over a region R in the xy plane.

$$\iint_R f(x,y)\,dA$$

base region. 2D

$\Rightarrow$ so have 2 $\int$

$\Longrightarrow$ dA = area of infinitesimal piece of the base region R.

Cf Calc ~~1~~ 2:

$$\int_a^b f(x)\,dx$$