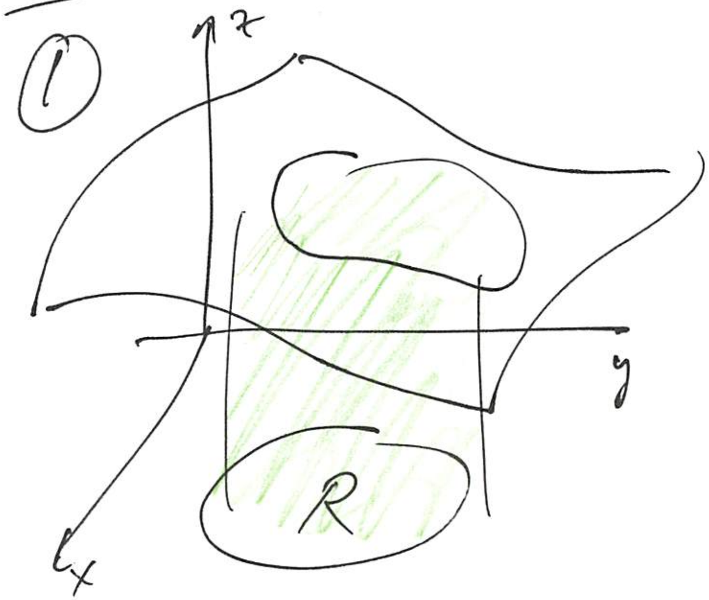
Multiple integrals

1. idea
2. iterated integrals over rectangular base regions
3. more general regions.

1.

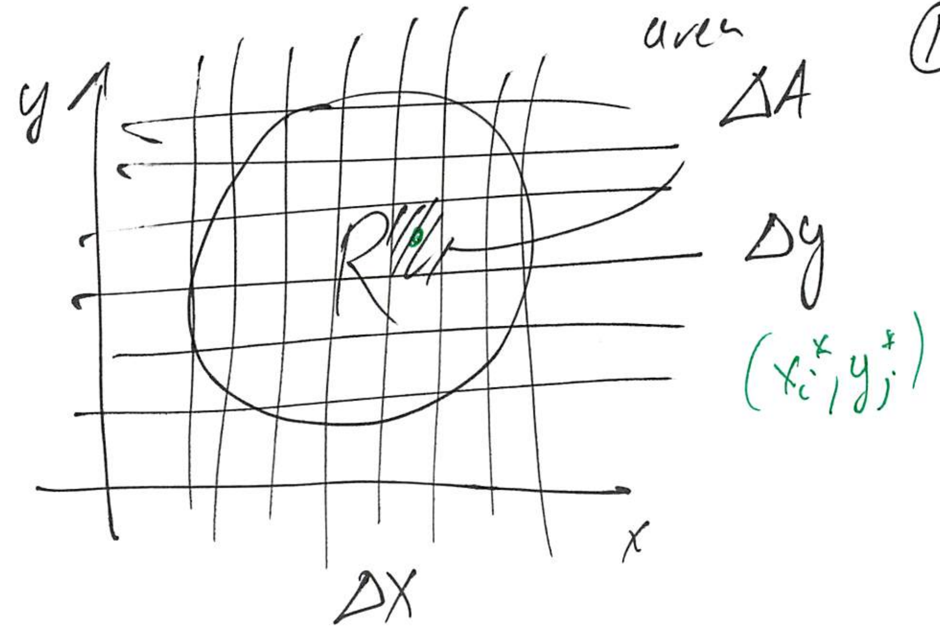


$z = f(x,y)$

$$\iint_R f(x,y)\,dA$$

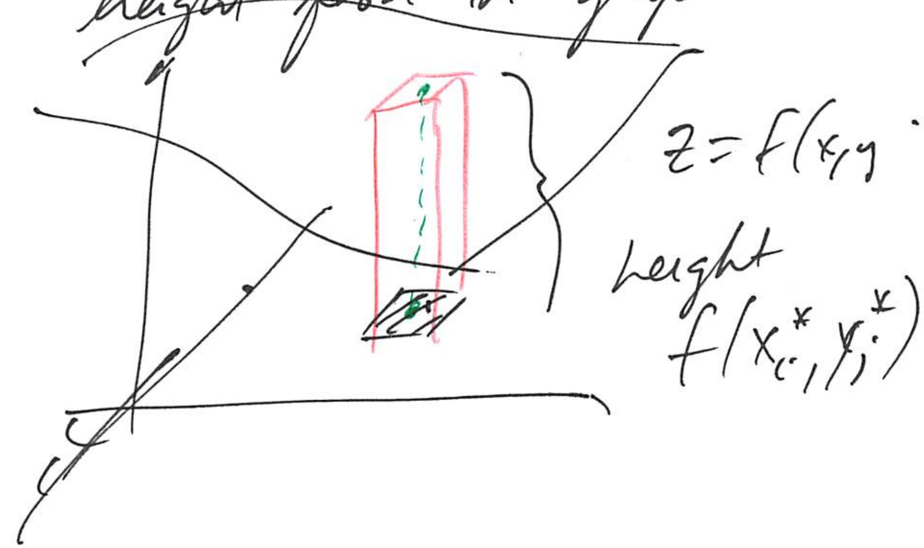
R base region

dA tiny piece of area in base



$\Delta A = \Delta X \Delta y$.

over each piece, build rectangular box by taking height from the graph.



where $(x_i^*, y_j^*) \in$ the tiny square

sample point.

Total volume $\approx \sum_i \sum_j f(x_i^*, y_j^*) \Delta A$

take limit as $\Delta A \to 0$

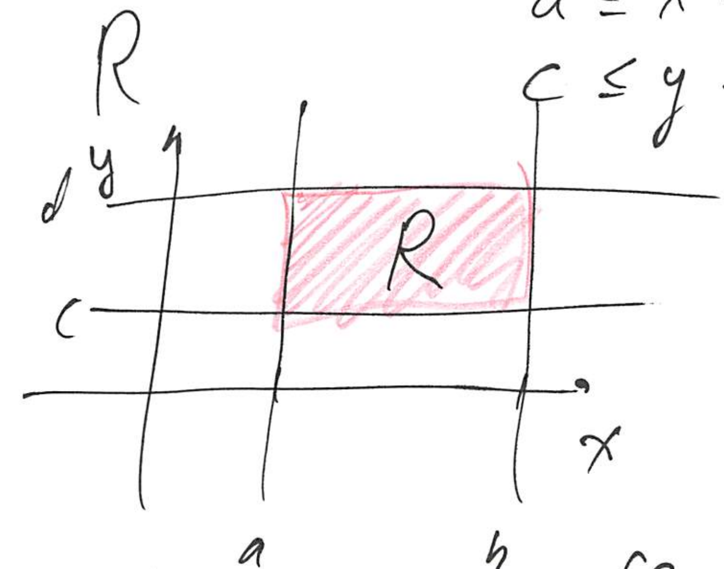
get $\iint_R f(x,y)\, dA$.

How to compute this?
do iterated integration.
Integrate with respect to one variable, keeping the other variable as a constant.
Then repeat.

2) 1st treat rectangular base regions

$a \le x \le b$

$c \le y \le d$



want to compute $\iint_R f(x,y)\, dA$.

1) choose ~~and~~ an order for x, y.

2. integrate with respect to the first variable, treating the other as a constant.
3. 1st var is now gone. integrate with respect to the 2nd variable.

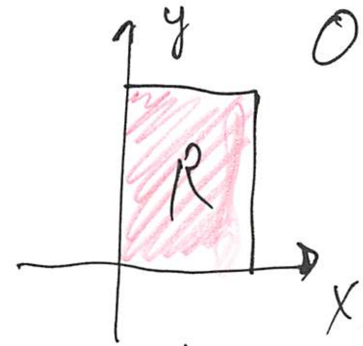
$\Delta A = \Delta x \Delta y = \Delta y \Delta x$

$\rightsquigarrow dA$ either $dx\,dy$ or $dy\,dx$

$$dA = \begin{cases} dx\,dy & x \text{ 1st}, y \text{ 2nd} \\ dy\,dx & y \text{ 1st}, x \text{ 2nd} \end{cases}$$

e.g. $f(x,y) = xy^2$

$R: \quad 0 \le x \le 1$
$\quad\;\; 0 \le y \le 2$



Want $\iint_R xy^2\,dA$

$\int \ldots dx$

1. x first, y second.
$dA = dx\,dy$

$$\iint_R xy^2\,dx\,dy = \int\int xy^2\,dx\,dy$$

$$= \int_0^2 \left(\int_0^1 x y^2 \, dx \right) dy$$

② do innermost 1st.

$$\int_0^2 y^2 \left(\int_0^1 x \, dx \right) dy$$

$$= \int_0^2 y^2 \left(\frac{1}{2} x^2 \Big|_0^1 \right) dy$$

③ $$= \int_0^2 y^2 \cdot \frac{1}{2} \, dy$$

$$= \frac{1}{6} y^3 \Big|_0^2 = \boxed{\frac{4}{3}}$$

④

Could do with the other order of integration:

$dA = dy\,dx$

$0 \le x \le 1$

$0 \le y \le 2$

$$\int_0^1 \int_0^2 x y^2 \, dy \, dx$$

$$= \int_0^1 x \left(\int_0^2 y^2 \, dy \right) dx$$

$$= \int_0^1 x \left(\frac{1}{3} y^3 \Big|_0^2 \right) dx$$

$$= \int_0^1 \frac{8x}{3} \, dx = \frac{8}{3} \cdot \frac{1}{2} x^2 \Big|_0^1$$

$$= \boxed{4/3}$$

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e.g. $\iint_R y \sin(xy)\, dA$

$1 \le x \le 2$

$0 \le y \le \pi$

order 1: $dA = dx\,dy$.

$$\int_0^\pi \int_1^2 y \sin xy \; dx\, dy$$

$$= \int_0^\pi \left(\int_y^{2y} \sin u \, du \right) dy \qquad \begin{array}{l} u = yx \\ du = y\,dx \end{array}$$

$$= \int_0^\pi \left(-\cos u \Big|_y^{2y} \right) dy$$

$$= \int_0^\pi (\cos y - \cos 2y)\, dy$$

$$= \sin y - \frac{1}{2} \sin 2y \Big|_0^\pi$$

$$= \boxed{0}.$$

order 2: $dA = dy\,dx$

$$\int_1^2 \int_0^\pi y \sin xy \; dy\, dx$$

need antiderivative, but w.r.t. y, not x.

need to do integration by parts

$$\int u\,dv = uv - \int v\,du$$

$u = y$ $\qquad$ $v = -\frac{1}{x}\cos(xy)$

$dv = \sin(xy)\,dy$ $\qquad$ $du = dy$

... get $\boxed{0}$

5 **more general base regions.**

again use technique of iterated integrals.

limits will now be more complicated.

e.g. R triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$

$$\iint_R f(x,y)\,dA = ?$$

right now focus is setting up the integration, not evaluating it.

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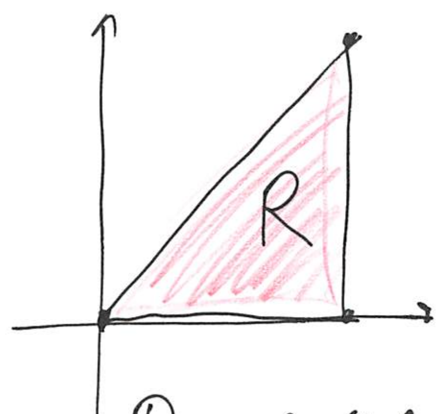
To set up integral, need to represent R using inequalities.

[cf. R rectangle: $a \le x \le b$, $c \le y \le d$]



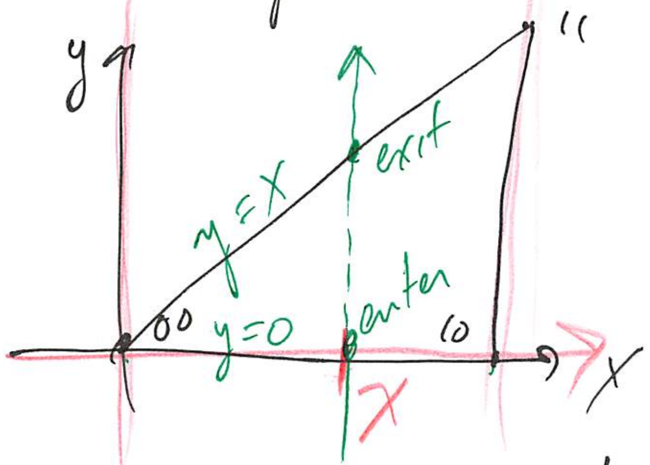
main point: limits of the inequalities will involve functions

The answer will have the form

$\# \le x \le \#$.

fns of x $\le y \le$ fns of x

How to find this description



$$\underline{0} \le x \le \underline{1}$$

enter $\underline{0} \le y \le \underline{x}$ exit

$\Rightarrow$ description of R using inequalities

$$0 \le x \le 1$$
$$0 \le y \le x$$

This description corresponds to order of integration

$$dA = dy\,dx$$

$$\int_{0\ \#}^{1\ \#} \int_{\text{fns of } x}^{\text{fns of } x} f(x,y)\,dy\,dx$$

$$\int_{0}^{1} \int_{0}^{x} f(x,y)\,dy\,dx$$

Take $f(x,y) = 1$.

$$\iint_R 1 \cdot dA = \text{area}(R)$$

$$\int_0^1\int_0^x dy\,dx = \int_0^1 y\Big|_0^x dx$$

$$= \int_0^1 x\,dx$$

$$= \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2} \checkmark$$

$= \text{area}(\Delta)$

Let's reverse the order of integration.

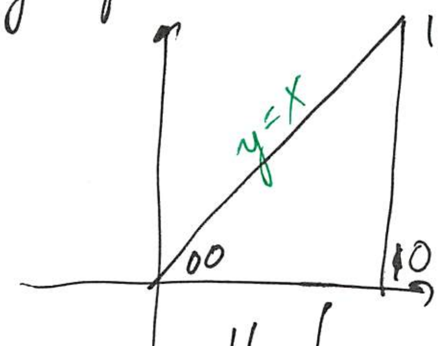
With rectangles, we just swap the numbers

$$\int_a^b\int_c^d \cdots \Rightarrow \int_c^d\int_a^b$$

Can't do this now!!

We must be more careful

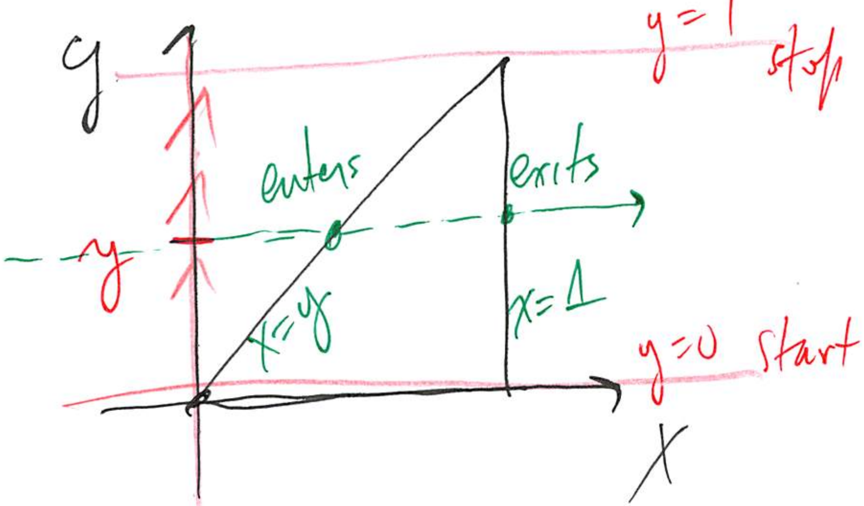
try for our triangular R.



$dA = dx\,dy$

$\Updownarrow$

| # | $\_\_ \le y \le \_\_$ |
| fns of y | $\_\_ \le x \le \_\_$ |



$$\underline{\;0\;} \le y \le \underline{\;1\;}$$

enter $\underline{\;y\;} \le x \le \underline{\;1\;}$ exit.

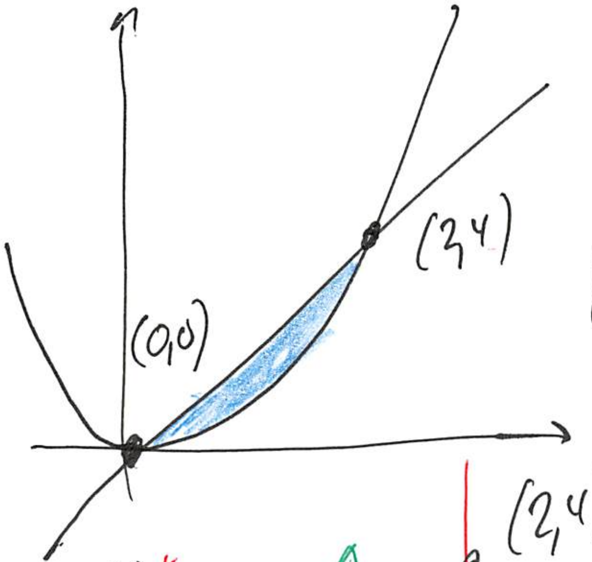
$$\Longrightarrow \int\int f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_y^1 f(x,y)\,dx\,dy$$

$$\int_0^1 \int_0^x f(x,y)\,dy\,dx = \int_0^1 \int_y^1 f(x,y)\,dx\,dy$$

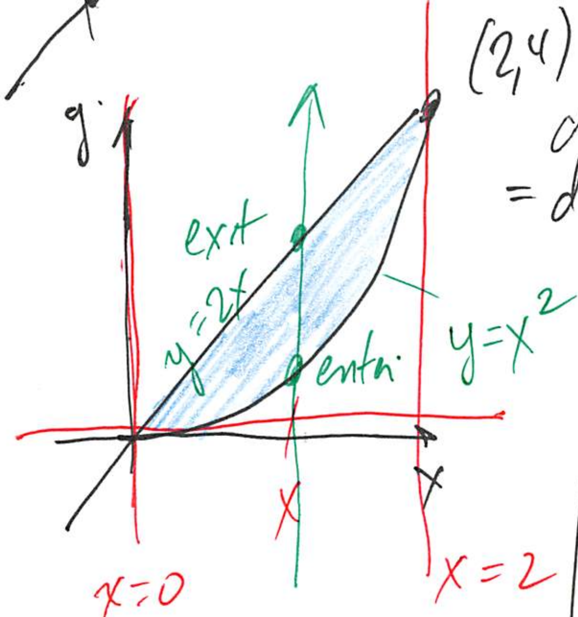
e.g. R = bounded region between graphs of $y = x^2$, $y = 2x$.

Write $\iint_R f\,dA$ as iterated integral. do both orders.



$x^2 = 2x$

R.

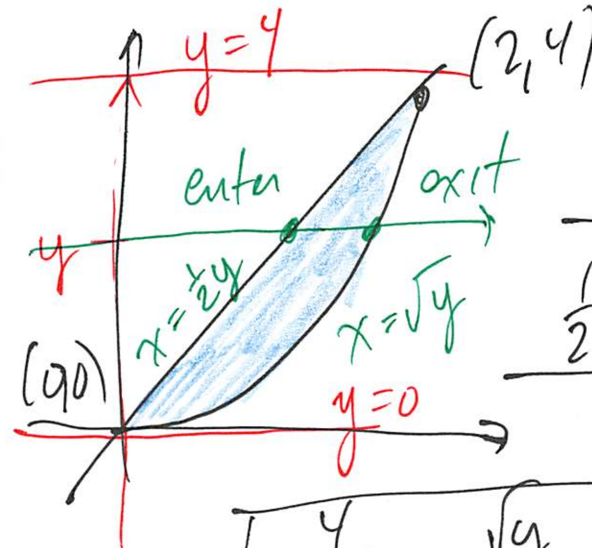


$dA = dy\,dx$

$0 \le x \le 2$

$x^2 \le y \le 2x$

$$\boxed{\int_0^2 \int_{x^2}^{2x} f \, dy \, dx}$$



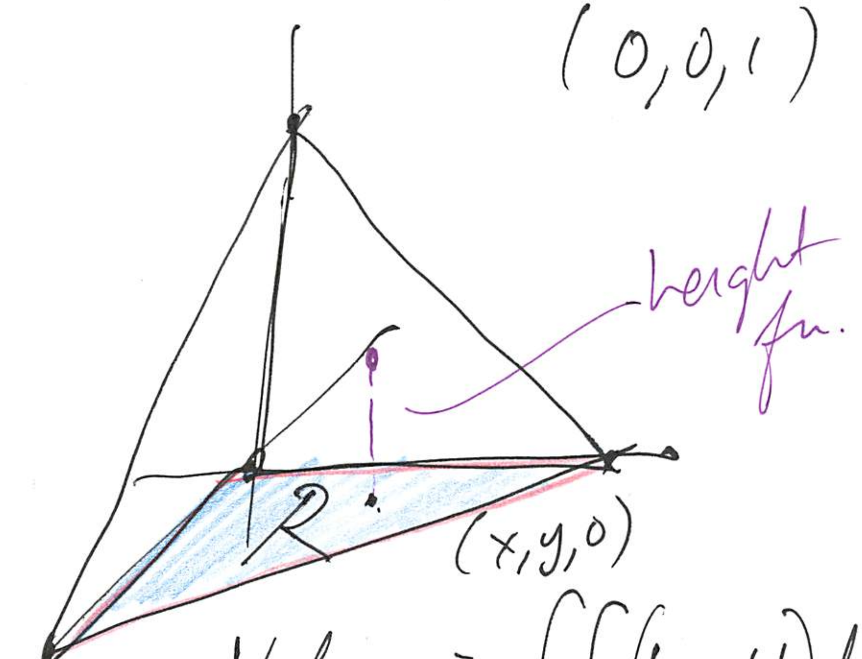
$0 \le y \le 4$

$\frac{1}{2}y \le x \le \sqrt{y}$

$$\boxed{\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} f \, dx \, dy}$$

e.g. volume of tetrahedron with verts

$(0,0,0), (1,0,0), (0,1,0)$
$(0,0,1)$



$$\text{Volume} = \iint_R (\text{height})\, dA$$

base region $\to R$

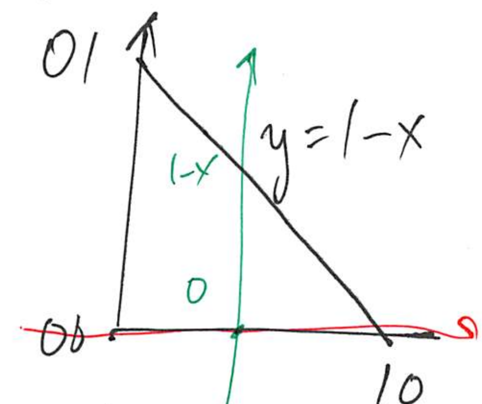
Front face:

$$x+y+z=1$$

$$z = 1-x-y$$

$$\iint_R (1-x-y)\, dA$$

Base region



$dA = dy\,dx \quad 0 \le x \le 1$

$0 \le y \le 1-x$

$$\int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

ans : $\boxed{\frac{1}{6}}$.