①

Last time: Line integral with respect to arc length.

C curve $\vec{r}(t)$ $a \le t \le b$

f function.

$\int_C f\, ds$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

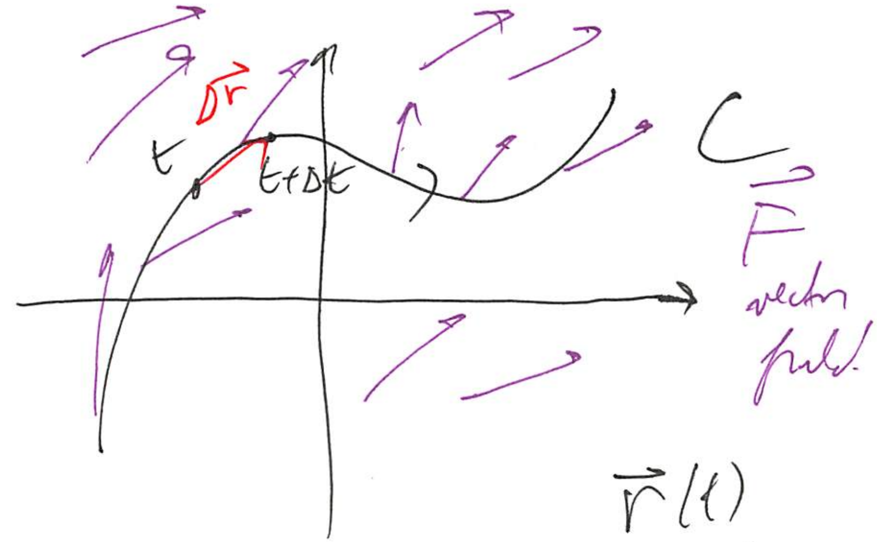
$= |\vec{r}\,'(t)|\, dt.$

Today: Line integral.

C curve, $\vec{F}$ vector field.

$\int_C \vec{F} \cdot d\vec{r}$

$d\vec{r}$ tiny vector increment along C.

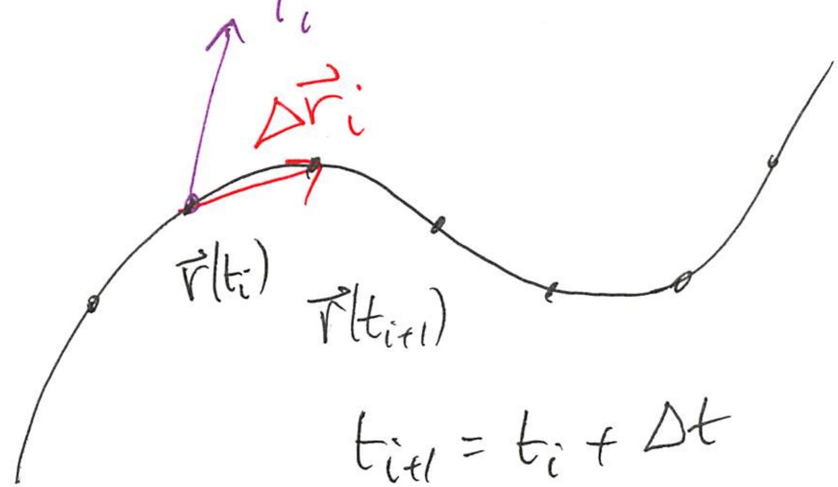


$\vec{r}(t)$

$a \le t \le b$

break time interval into small pieces Δt

get small vector pieces $\Delta\vec{r}$



$t_{i+1} = t_i + \Delta t$

approximation is

$$\sum_i \vec{F}_i \cdot \Delta \vec{r}_i$$

Take limit as $\Delta t \to 0$,

get $$\int_C \vec{F} \cdot d\vec{r}$$

application: "Work" from physics.

Output related to effort expended to undergo motion along C in the presence of force field $\vec{F}$.

To compute, convert to a usual integral in t.

$$\frac{d\vec{r}}{dt} = \text{velocity} = \vec{r}\,'(t)$$

$$d\vec{r} = \vec{r}\,'(t)\,dt.$$

on C $\vec{F}(x,y) = \vec{F}(x(t), y(t))$

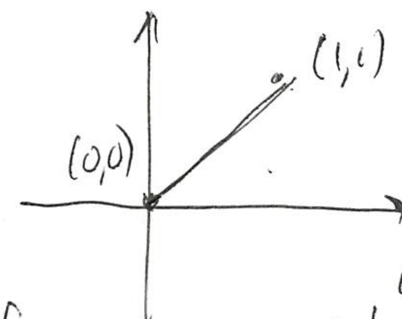
where $\vec{r}(t) = \langle x(t), y(t) \rangle$.

e.g. $\vec{F} = \langle x, y \rangle$.

C = line segment

$\vec{r}(t) = \langle \overset{x(t)}{t}, \overset{y(t)}{t} \rangle$

$0 \le t \le 1$



$d\vec{r} = \langle 1, 1 \rangle dt$

on C, $\vec{F} = \langle t, t \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 2t\,dt = t^2\Big|_0^1 = \boxed{1}$$

e.g. $\vec{G} = \langle -y, x \rangle$.

ⓐ $\int_{C_1} \vec{G} \cdot d\vec{r}$ $\quad C_1 =$ prev problem

ⓑ $\int_{C_2} \vec{G} \cdot d\vec{r}$ $\quad C_2$ parabola $y = x^2$.

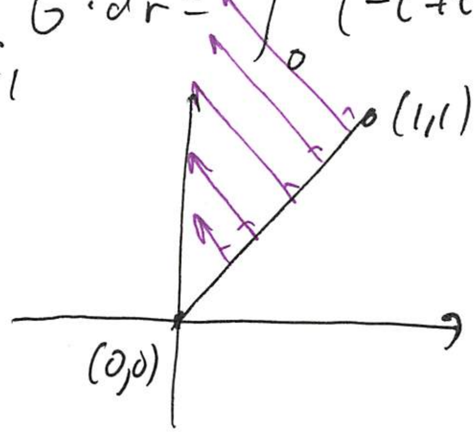
$\langle t, t^2 \rangle \quad 0 \le t \le 1$

ⓐ $\vec{r}(t) = \langle t, t \rangle$

$\vec{r}\,'(t) = \langle 1, 1 \rangle$

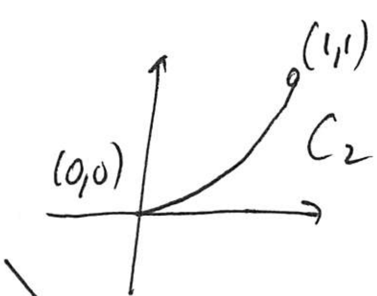
$\vec{G} = \langle -y, x \rangle = \langle -t, t \rangle$ on C_1

$$\int_{C_1} \vec{G} \cdot d\vec{r} = \int_0^1 (-t+t)\,dt = \boxed{0}$$



③

ⓑ $\int_{C_2} \vec{G} \cdot d\vec{r}$



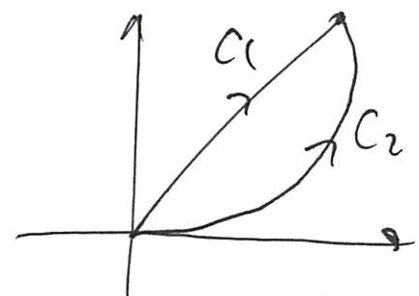
$\vec{r}(t) = \langle t, t^2 \rangle$

$d\vec{r} = \langle 1, 2t \rangle\,dt$

$$\int_0^1 \langle -t^2, t \rangle \cdot \langle 1, 2t \rangle\,dt$$

$$= \int_0^1 (-t^2 + 2t^2)\,dt = \int_0^1 t^2\,dt$$

$$= \frac{1}{3}t^3 \Big|_0^1 = \boxed{\frac{1}{3}}$$



$\Longrightarrow$ In general output depends on the path, not just end pts.

Comments

1. another version of the notation is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

$$\vec{F} = \langle P(x,y), Q(x,y) \rangle$$

$$\vec{r} = \langle x(t), y(t) \rangle$$

$$\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

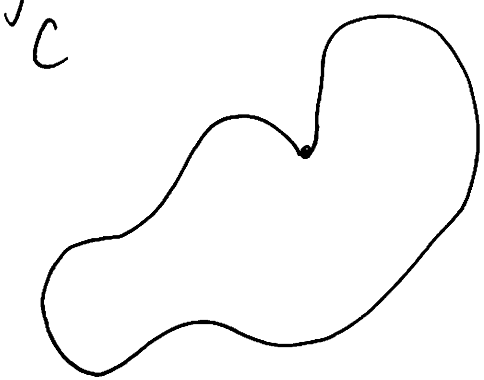
$$d\vec{r} = \langle dx, dy \rangle$$

e.g. 2nd example ($\vec{G}$)

$$\int_C -y\,dx + x\,dy$$

2. Special notation if the curve is closed, meaning the starting and ending pts coincide. we write

$$\oint_C \vec{F} \cdot d\vec{r}$$



Fundamental theorem for line integrals

generalization of the fundamental theorem of calculus.

Vector fields that are relevant are called <u>conservative vector fields</u>.

<u>Def</u> $\vec{F}$ is conservative if

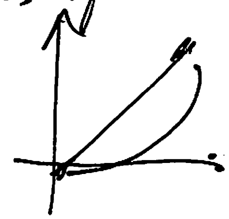
$$\int_C \vec{F}\cdot d\vec{r}$$

depends only on the endpoints of C, not on the path between them.

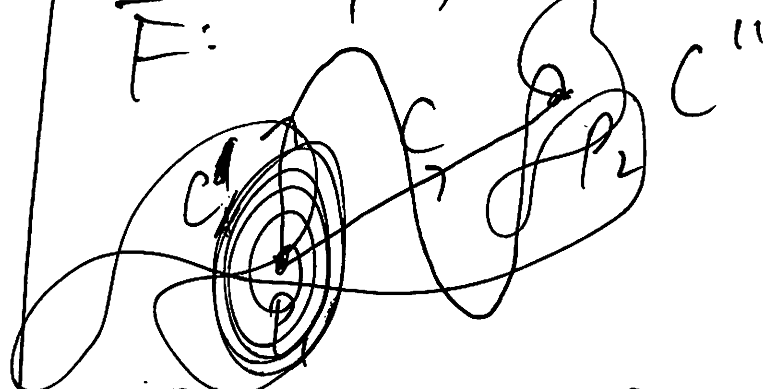
e.g. $\vec{F} = \langle x, y\rangle$ is conservative.

$\vec{G} = \langle -y, x\rangle$ is not conservative.

(got different answers for $\int_{C_1}$, $\int_{C_2}$).



$\vec{F}$:



$$\int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r} = \int_{C''} \vec{F}\cdot d\vec{r}$$

Q: how do you tell if conservative?

A: If $\vec{F} = \nabla f$, and there are no missing points in the domain of the vector field, then $\vec{F}$ is conservative

eg $\vec{F} = \langle x, y \rangle$

$= \nabla f$ where

$f = \frac{1}{2}(x^2 + y^2)$

$\Rightarrow$ conservative

eg $\vec{G} \neq \nabla g$ for any ~~such~~ g.

$\Rightarrow$ not conservative

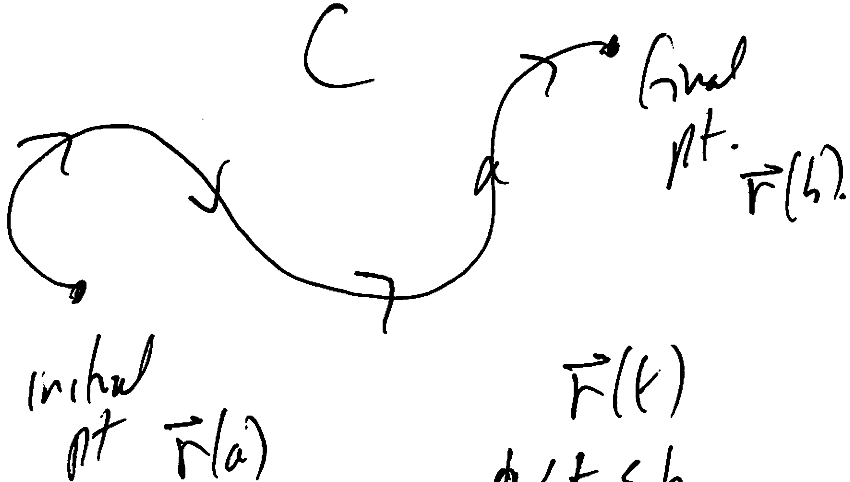
Def If $\vec{F} = \nabla f$, then f is called a potential function for $\vec{F}$

analogue of an antiderivative of $\vec{F}$

Thm Fundamental thm of line integrals.

Suppose $\vec{F}$ is conservative. $\vec{F} = \nabla f$. Then

$$\int_C \vec{F} \cdot d\vec{r} = f\left(\begin{smallmatrix}\text{final}\\ \text{pt}\end{smallmatrix}\right) - f\left(\begin{smallmatrix}\text{initial}\\ \text{pt}\end{smallmatrix}\right)$$



$\vec{r}(t)$

$a \le t \le b$

compare with usual fund thm.

$$\int_C \vec{F} \cdot d\vec{r} = f(\text{final}) - f(\text{initial})$$

$$\nabla f = \vec{F}$$

$$\int_a^b g(x)\,dx = G(b) - G(a)$$

$$G'(x) = g.$$

Q:

① How to tell if $\vec{F}$ is conservative?

② If so, how to find potential fn f?

A:

Suppose $\vec{F} = \langle P, Q \rangle$

If $\vec{F} = \nabla f$, then

$P = f_x$, $Q = f_y$

need $f_{xy} = f_{yx} \Longrightarrow \boxed{P_y = Q_x}$

Gives condition for 2D vector fields

for 3D, $\vec{F} = \nabla f$
$\qquad\qquad\quad \langle P, Q, R \rangle$

$P = f_x$, $Q = f_y$, $R = f_z$
and equality of all possible mixed partials.

$$f_{xy} = f_{yx} \qquad \boxed{P_y = Q_x}$$
$$f_{xz} = f_{zx} \qquad \boxed{P_z = R_x}$$
$$f_{yz} = f_{zy} \qquad \boxed{Q_z = R_y}$$

(1) How to find f?

Ans: integrate!

$\vec{F} = \langle x, y \rangle$ $\quad P = x$, $Q = y$

$P_y = 0 = Q_x$ ✓

$\Longrightarrow$ conservative.

$P = f_x = x$
So need f to satisfy

$$\frac{\partial f}{\partial x} = x$$

$$\int f_x \, dx = \int x \, dx = \frac{1}{2}x^2 + C$$

really any function of y.

better: $\frac{1}{2}x^2 + C(y)$

have so far $f(x,y) = \frac{1}{2}x^2 + C(y)$

also know

$$\frac{\partial f}{\partial y} = y$$

$$\frac{\partial f}{\partial y} = C'(y) = y.$$

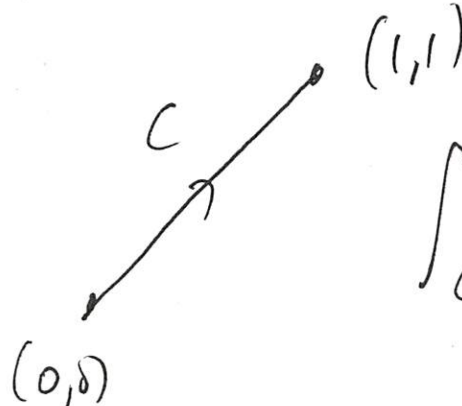
$$\int y\,dy = \frac{1}{2}y^2 + C$$

now just a number.

Ans:

$$f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$$

Can just take $C = 0$ unless other info is present.



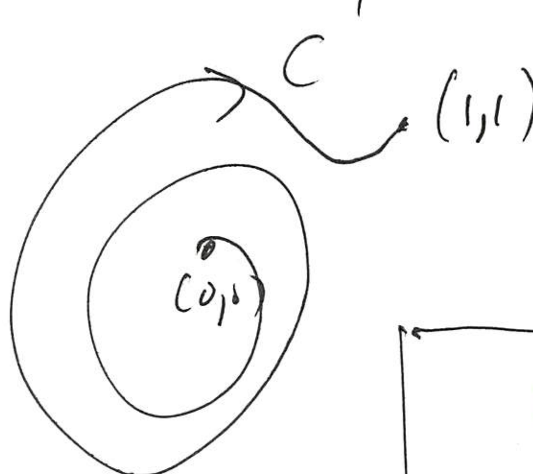
$$\int_C \vec{F}\cdot d\vec{r} = ?$$

|| Fund thm

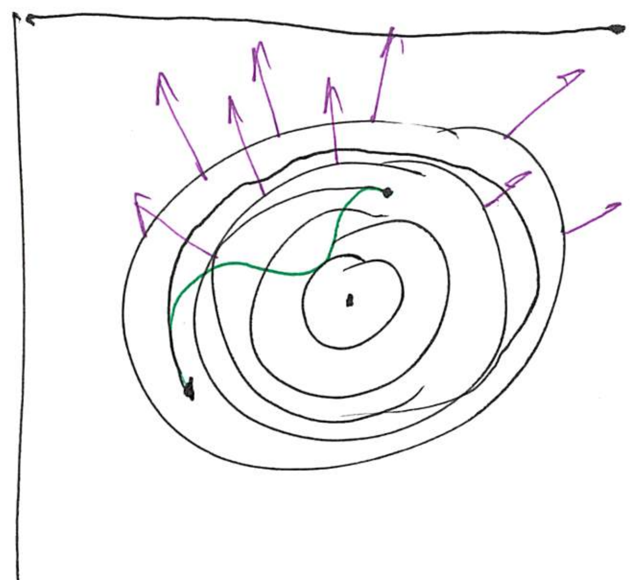
$$f(1,1) - f(0,0)$$

$$f = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

$$= 1 - 0 = 1$$



still get 1.



e.g. find potential fn for

$\vec{F} = \langle x^2 + xy,\ y^3 + \frac{1}{2}x^2 \rangle$

$\qquad\quad P \qquad\qquad Q$

conser: $P_y = Q_x$? ✓

$\qquad\ \ \overset{\shortparallel}{x} \quad\ \ \overset{\shortparallel}{x}$

$f_x = x^2 + xy \ , \ f_y = y^3 + \frac{1}{2}x^2$

$f = \int \left(y^3 + \frac{1}{2}x^2\right) dy =$

(using f_y) $\frac{1}{4}y^4 + \frac{1}{2}x^2 y + C(x)$

now use f_x.

$\frac{\partial}{\partial x}\left(\frac{1}{4}y^4 + \frac{1}{2}x^2 y + C(x)\right) =$

$xy + C'(x) \overset{?}{=} x^2 + xy$

$C' = x^2$

$C = \frac{1}{3}x^3$

$\Longrightarrow f = \frac{1}{4}y^4 + \frac{1}{2}x^2 y + \frac{1}{3}x^3$