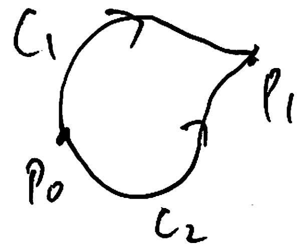
Last time:

- Line integrals $\int_C \vec{F} \cdot d\vec{r}$

- conservative vector fields



Line integrals are path independent

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

- Fund. thm of line integrals

Suppose $\vec{F}$ conservative

Let $\vec{F} = \nabla f$ (potential function)

$$\int_{C_1} (\nabla f) \cdot d\vec{r} = f(P_1) - f(P_0)$$

Today: Green's thm ①

Another version of fund thm.

Enables you to compute

$$\oint_C \vec{F} \cdot d\vec{r}$$

where C closed curve

$\vec{F}$ not conservative.

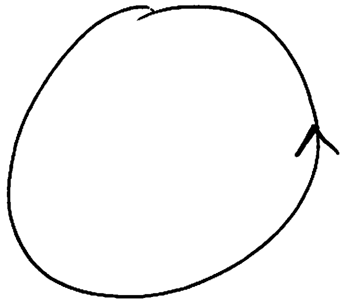
Remark If $\vec{F}$ is conservative,

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

$P_0 = P_1$ because C is closed

$\Longrightarrow f(P_0) = f(P_1)$

assume C is a <u>simple</u> <u>closed</u> curve. simple means no self-intersections!

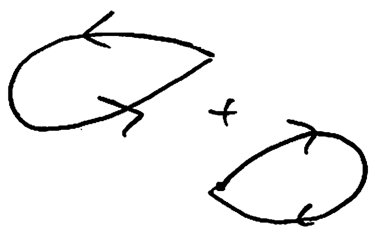


✓

C

C'

X

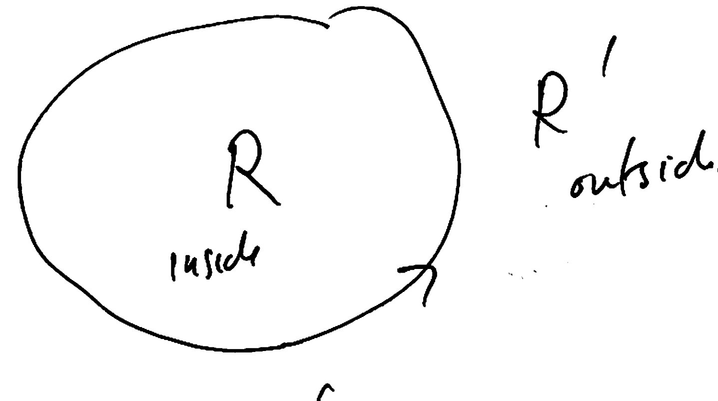


C_1 + C_2

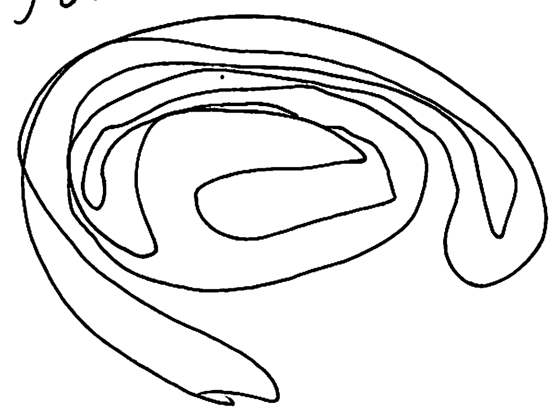
✓ ✓

i.e. we break C' into 2.

Fact: any simple closed curve divides the plane into 2 regions:



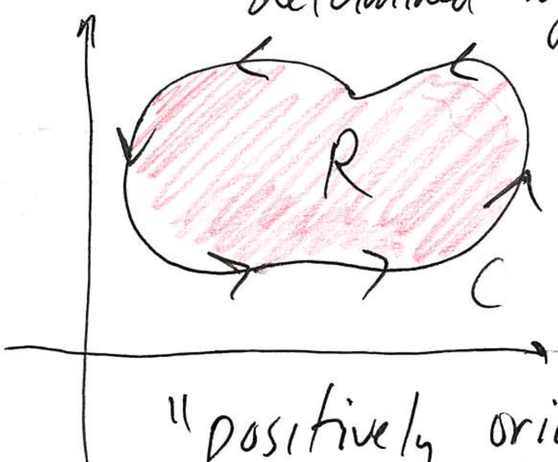
One region will be bounded.
"Jordan curve theorem"



Green's Thm: Line integral is connected to a double integral.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \dots\, dA$$

R = the bounded region determined by C.



assume C is "positively oriented"

"positively oriented" means R is on the left as you traverse C.

$$\vec{F} = \langle P(x,y), Q(x,y) \rangle$$

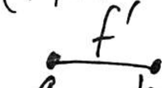
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$

Green's Thm:

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA}$$

Claim: version of a Fund. thm.

Calc 2: $\int_a^b f'(x)\,dx = f(b) \underset{f}{-} f(a)$



L.I.: $\int_C \nabla f\cdot d\vec{r} = f(p_1) \underset{f}{-} f(p_0)$

$$\iint_R (Q_x - P_y)\,dA = \oint_C P\,dx + Q\,dy$$

$Q_x - P_y$ $\qquad\qquad$ $\langle P, Q \rangle$

$L \Rightarrow R$ dim of region goes down by 1.

$R \Rightarrow L$ something gets differentiated.

Remark: remember that to test whether $\vec{F}$ is conservative, you check $Q_x \overset{?}{=} P_y$

$\vec{F}$ conser. $\Rightarrow \iint_R 0\,dA$

e.g. $\vec{F} = \langle -y, x \rangle = \langle P, Q \rangle$.

C unit circle, oriented counterclockwise.

compute both sides of G.T.

① $\oint \vec{F} \cdot d\vec{r} = ?$

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$0 \le t \le 2\pi$

$d\vec{r} = \langle -\sin t\,dt, \cos t\,dt \rangle$

on C, $\vec{F} = \langle -\sin t, \cos t \rangle$

$$\oint \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= \int_0^{2\pi} dt = \boxed{2\pi}!$$

② $\iint_R (Q_x - P_y)\,dA$

$= \iint_R (1-(-1))\,dA = 2\iint_R dA$

$= 2 \times \text{area}(R)$

$= \boxed{2\pi}$

✓

e.g. $\oint_C x^4\,dx + xy\,dy$

$= \int_{C_1} \cdots + \int_{C_2} \cdots + \int_{C_3} \cdots$

use G.T.

③ $\iint_R (Q_x - P_y)\,dA$

$P = x^4 \qquad Q = xy$

$P_y = 0 \qquad Q_x = y.$

$\|$

$\iint_R y\,dA. \qquad R: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{cases}$

$\int_0^1 \int_0^{1-x} y\,dy\,dx = \boxed{\dfrac{1}{6}}$

e.g. C = unit circle counterclockwise

$\oint_C \underbrace{(3y - e^{\sin x})}_{P}\,dx + \underbrace{(7x + \tan y)}_{Q}\,dy = ?$

$P = 3y - e^{\sin x}$ $\quad Q = 7x + \tan y$

$P_y = 3$ $\quad Q_x = 7$

$$\iint_R (7-3)\, dA = 4 \iint_R dA = \boxed{4\pi}$$

Remark:

$$\vec{F} = \vec{E} + \vec{G} = \langle 3y, 7x \rangle + \langle -e^{\sin x}, \tan y \rangle$$

$\vec{E}$ not conservative

$\vec{G}$ is conservative

$$\Rightarrow \oint \vec{G} \cdot d\vec{r} = 0$$

e.g. we can use the line integral to compute the double integral.

e.g. $\oint_C \frac{1}{2}(x\, dy - y\, dx)$

$= \text{area of } R.$

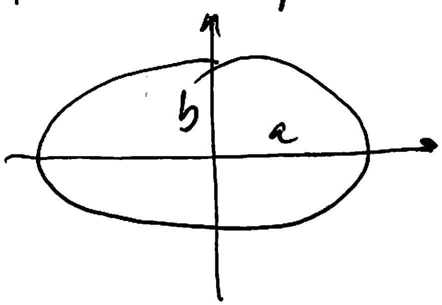
$P = -\frac{1}{2} y$ $\quad Q = \frac{1}{2} x$

$P_y = -\frac{1}{2}$ $\quad Q_x = \frac{1}{2}$

$Q_x - P_y = 1$

$$\oint_C \frac{1}{2}(x\,dy - y\,dx) = \iint_R 1\,dA = \text{area}(R).$$

Try R = ellipse.



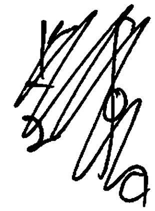
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$0 \le t \le 2\pi$

$x = a\cos t$
$y = b\sin t$

$dx = -a\sin t\,dt$
$dy = b\cos t\,dt$

$\Rightarrow$



$$\frac{1}{2}\int_0^{2\pi} ab\cos^2 t\,dt + ab\sin^2 t\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt = \boxed{\pi ab}!$$

Curl, divergence of a vector field.
versions of the derivative for vector fields

∇, curl, divergence
gradient

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	Input	Output
∇ :	function of 3 variable $f(x,y,z)$	$\Longrightarrow$ vector field $\nabla f = \langle f_x, f_y, f_z \rangle$
curl :	vector field in 3D $\vec{F}$	$\Longrightarrow$ vector field in 3D; $\vec{F} \longmapsto \text{curl}\,\vec{F}$
div :	vector field in 3D $\vec{F}$	$\Longrightarrow$ function of 3 vars; $\vec{F} \longmapsto \text{div}\,\vec{F}$

curl

$$\vec{F} = \langle P, Q, R \rangle$$

curl $\vec{F}$ = new v.f. built from the derivatives of P, Q, R.

Notation:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

"operator" $= \langle \partial_x, \partial_y, \partial_z \rangle$.

$$\partial_x P := \frac{\partial P}{\partial x}$$

$\vec{F} = \langle P, Q, R \rangle$

$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$

$$= \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix}$$

$$= \langle \partial_y R - \partial_z Q, \ \partial_z P - \partial_x R, \ \partial_x Q - \partial_y P \rangle$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

e.g. $\vec{F} = \langle xz, xyz, y^2 \rangle$

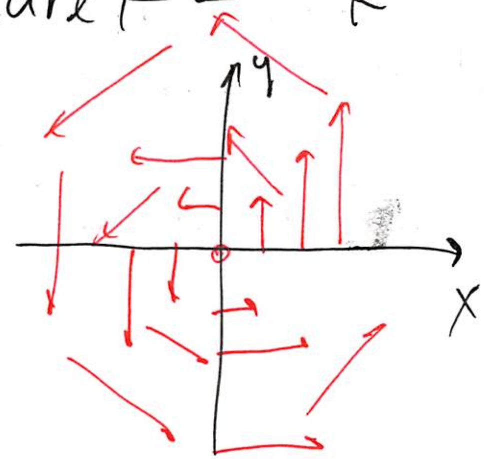
$\operatorname{curl} \vec{F} = \langle 2y - xy, x, yz \rangle$

Why <u>curl</u>?

$\operatorname{curl} \vec{F}$ has to do with rotational motion in $\vec{F}$.

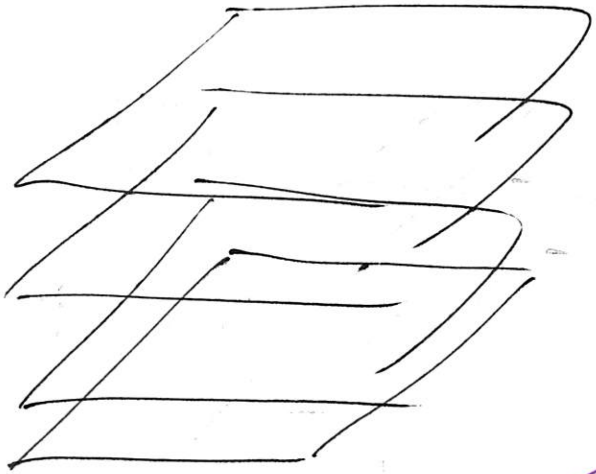
e.g. $\vec{F} = -\frac{1}{2} y \hat{\imath} + \frac{1}{2} x \hat{\jmath}$

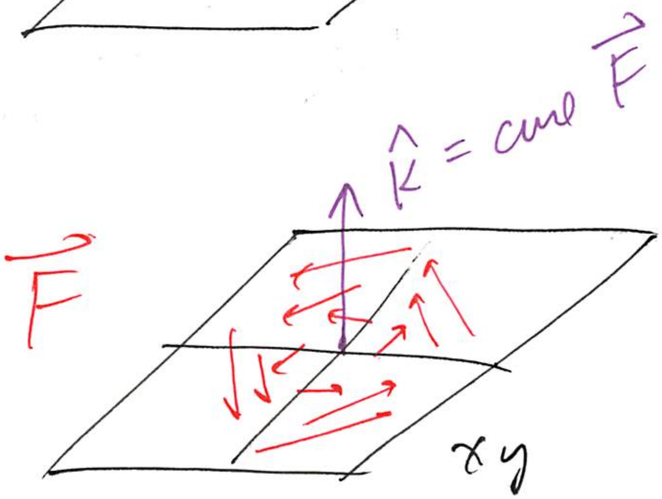
$\operatorname{curl} \vec{F} = \hat{k}$



$\vec{F}$ in xy plane.

to make 3D image, just stack this up along the z axis





$\vec{F}$ rotates around the z-axis

curl $\vec{F}$ points along the axis of rotation, direction is determined using right hand rule.