A TRANSFER MATRIX PROBLEM

This is really a mix of literature search and solving problems. It's just for fun. Consider the sequence of matrices M_i, $i = 1, 2, 3, \ldots$ given by

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

etc. You can see that they are built from binomial coefficients in a certain way.

The matrix M_1 is the adjacency matrix of a certain digraph, and we showed that the generating function $F_{11}(x)$ counting the walks starting and stopping at vertex 1 is the same as the generating function for the Fibonacci numbers $\{F_n\}$.

(1) Show that the generating function $F_{11}(x)$ built using the matrix M_k is that for the k-th power of the Fibonaccis $\{F_n^k\}$. For instance for M_2 you get

$$\begin{aligned} F_{11}(x) &= \frac{-x+1}{x^3-2x^2-2x+1} \\ &= 1 + x + 4x^2 + 9x^3 + 25x^4 + 64x^5 + 169x^6 + 441x^7 + 1156x^8 + 3025x^9 + \ldots \end{aligned}$$

One way to do this is to think about how to get the expression $(A\alpha^n + B\beta^n)^k$ to show up via a GF, but there is probably a direct way using the binomial coefficients. There are many crazy identities for Fibonacci numbers ...

(2) Here are the next few rational functions we get:

$$\frac{-x^2-2x+1}{x^4+3x^3-6x^2-3x+1}, \frac{-x^3+4x^2+4x-1}{x^5-5x^4-15x^3+15x^2+5x-1},$$
$$\frac{-x^4-7x^3+16x^2+7x-1}{x^6+8x^5-40x^4-60x^3+40x^2+8x-1},$$
$$\frac{-x^5+12x^4+53x^3-53x^2-12x+1}{x^7-13x^6-104x^5+260x^4+260x^3-104x^2-13x+1}.$$

What is the pattern? (OEIS is a big help here. Also you can check out Knuth *Art of Computer Programming vol 1*, problems 29, 30 in §1.2.8.)

(3) For M_1 we were able to interpret walks on the digraph starting and stopping at vertex 1 in terms of tiling a $1 \times n$ rectangle with 1×1 and 1×2 tiles, or in terms of arranging 1 ft and 2 ft flags on a flagpole. What is the walks interpretation for M_k? It may involve tiles, flags, compositions, ... I don't know a good answer for $k \geq 3$. People have given various interpretations of these numbers, but can you make them correspond to the walks?