## MATH 233H ATTENDANCE PROBLEMS

These are the quick problems given in class to (randomly) take attendance. Please let me know if you find any mistakes.

(1) Let $\mathbf{v} = \langle 1, 1, 1\rangle, \mathbf{w} = \langle 1, 3, -1\rangle$. Find a unit vector perpendicular to $\mathbf{v}$ and $\mathbf{w}$. **Answer:** Take the cross product to get $\mathbf{r} = \langle -4, 2, 2\rangle$. Using the dot product we see that $\mathbf{r}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. The length of $\mathbf{r}$ is $\sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$. So the unit vector is $\langle -2, 1, 1\rangle/\sqrt{6}$.

(2) Find the distance between the lines $\langle t, -t, 1\rangle$ and the $x$-axis. **Answer:** The lines are skew. The direction vectors are $\mathbf{i}$ and $\langle 1, -1, 0\rangle = \mathbf{i} - \mathbf{j}$, so a common perpendicular is $\mathbf{k}$. A vector going in between is $\mathbf{w} = \langle 0, 0, 1\rangle$ (take $t = 0$ and the origin) so the distance is $|\mathbf{w} \cdot \mathbf{k}| = 1$.

(3) Let $\mathbf{r}(t) = \langle te^t, t^2 + t, \sin t^2\rangle$. Compute $\mathbf{r}'$ and $\mathbf{r}''$. **Answer:** $\mathbf{r}' = \langle te^t + e^t, 2t + 1, 2t\cos t^2\rangle$, $\mathbf{r}'' = \langle 2e^t + te^t, 2, -4t^2 \sin t^2 + 2\cos t^2\rangle$.

(4) Let $f(x, y) = \cos(\sqrt{x^2 + y^2})$. (a) Draw some contour lines for $f$. (b) Describe/sketch the graph. **Answer:** Let $r = \sqrt{x^2 + y^2}$. Then $r$ measures the distance to the origin in the $xy$-plane, and the function is really $z = \cos r$ (Figure 1). This means the contour lines are circles centered at the origin (why?). The graph of the surface is what you get when you rotate the graph of cosine about the vertical axis (see Figure 2, although it's somewhat distorted). If you want to plot it yourself, you can try `plot cos sqrt (x x + y y)` at `wolframalpha.com`.

(5) Let $f(x, y) = x^3 + y^2 + x^2y$. Compute the first and second partial derivatives. Show that the mixed partials are equal. **Answer:** $f_x = 3x^2 + 2xy$, $f_y = 2y + x^2$, $f_{xx} = 6x + 2y$, $f_{yy} = 2$, $f_{xy} = 2x$, $f_{yx} = 2x$, and the mixed partials are equal.

(6) Let $f(x, y) = xy \sin(x^2 + y^2)$. Find the rate of change of $f$ at the point $(1, 2)$ when moving towards the point $(2, 3)$. **Answer:** This is asking for the directional derivative when you move from the point $(1, 2)$ in the direction headed towards $(2, 3)$. That is in the direction of the vector $\langle 2, 3\rangle - \langle 1, 2\rangle$ which is $\langle 1, 1\rangle$. To get the direction vector we have to convert this to a unit vector, so we get $\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$. We take the dot product of this with the gradient at $(1, 2)$, which is $\langle 2x^2y\cos(x^2 + y^2) + y\sin(x^2 + y^2), 2xy^2\cos(x^2 + y^2) + x\sin(x^2 + y^2)\rangle|_{(1,2)} = \langle 4\cos 5 + 2\sin 5, 8\cos 5 + \sin 5\rangle$. So the dot product is $(12\cos 5 + 3\sin 5)/\sqrt{2}$.

*Date*: December 5, 2025.

(7) Find the volume under graph of $xe^{xy}$ and over the rectangle $0 \leq x, y \leq 1$. **Answer:** We compute

$$\int_0^1 \int_0^1 xe^{xy}\, dy\, dx.$$

We do this order to avoid integration by parts. Then we get $\int_0^1 e^x - 1\, dx = e^x - x|_0^1 = e - 2$.

(8) Write the integrals needed to compute the centroid of the unit disk $x^2 + y^2 \leq 1$. **Answer:** We need the two moment integrals and the area integral. (For centroid, we compute the center of mass assuming the density $\rho$ is 1.) We use polar and write the region as $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$ and use $dA = r\, dr\, d\theta$. The integrands we need are (1) area $A$: $dA$; (2) moment about the $y$-axis $M_y : x\, dA = r\cos\theta\, dA$; and (3) moment about the $y$-axis $M_x : y\, dA = r\sin\theta\, dA$.

(9) Let $E$ be the region under the paraboloid $z = 7 - x^2 - y^2$, over the $xy$-plane, and inside the planes $x = 1$, $x = -1$, $y = 1$, $y = -1$. Write the integrals needed to compute the centroid (there are 4 of them). Do not evaluate the integrals. **Answer:** The region $E$ can be described by $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $0 \leq z \leq 7 - x^2 - y^2$. We take the integration order $dV = dz\, dy\, dx$. Then the 4 integrals each have the form

$$\int_{-1}^1 \int_{-1}^1 \int_0^{7-x^2-y^2} f\, dV.$$

The 4 different integrands are $f = 1$ (for the mass, since the density is 1) and $f$ respectively set to $x$, $y$, $z$ (for the moments with respect to the $yz$, $xz$, $xy$ planes).

(10) Compute $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy$. **Answer:** Best to use cylindrical coordinates. The base is a 1/4 unit disk, and the height goes between a paraboloid and a cone. The answer is 1/30.

(11) What is your favorite TV series? **Answer:** Breaking Bad was 1st. Tied for 2nd: Suits, The Good Place, Brooklyn 99.

(12) Draw level curves and the vector field for $\nabla f$, where $f = x^2 - y^2$. **Answer:** A picture is in the scanned notes on the website. The level curves are hyperbolas, and the gradient vector field is perpendicular to them, pointing in the direction of steepest ascent at every point.

(13) Let $\mathbf{F} = \langle x - y, x + y\rangle$ and $C$ be the unit circle, traversed once in the clockwise direction. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$. **Answer:** Use the standard parametrization of $C$ but be sure that you are going in the right direction. The value is $-2\pi$.

(14) Let $\mathbf{F} = \langle 2xy + z\sin(xz), x^2, 3 + x\sin(xz)\rangle$ and let $C$ be the unit circle in the $xy$-plane, traversed once in the counterclockwise direction about the $z$-axis (so going from $(1, 0, 0)$ in the direction of

$(0,1,0))$. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$. **Answer:** The vector field is conservative (check by computing all the relevant derivatives, or better check that $\operatorname{curl} \mathbf{F} = \mathbf{0}$). So the value is 0 since $C$ is a closed path.

(15) What is the best pizza topping? Justify your answer. **Answer:** The correct answer was not given and all justifications were not convincing.

(16) Let $S$ be the paraboloid $z = 1 - x^2 - y^2$ over the $xy$-plane. Represent $S$ as a parametric surface. Find $\mathbf{r}_u$ and $\mathbf{r}_v$ and their cross product. Compute $dS$. **Answer:** Take $\mathbf{r}(u,v) = \langle u, v, 1 - u^2 - v^2 \rangle$. The domain for $u, v$ is the unit disk $u^2 + v^2 \leq 1$ in the $uv$-plane. Then $\mathbf{r}_u = \langle 1, 0, -2u \rangle$, $\mathbf{r}_v = \langle 0, 1, -2v \rangle$, $\mathbf{r}_u \times \mathbf{r}_v = \langle 2u, 2v, 1 \rangle$. Finally $dS = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \sqrt{1 + 4u^2 + 4v^2} \, du \, dv$.

(17) Let $S$ be the paraboloid $z = 1 - x^2 - y^2$ over the $xy$-plane, oriented up. Let $\mathbf{F}$ be the vector field $\langle 0, 0, z \rangle$. Set up the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$. **Answer:** We use the parametrization from the previous problem. $d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v \, du \, dv = \langle 2u, 2v, 1 \rangle \, du \, dv$ and this points up. Then $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S z \, du \, dv = \iint_S (1 - u^2 - v^2) \, du \, dv$. This is best done by switching to polar coordinates in the $uv$-plane. The final integral is $\int_0^1 \int_0^{2\pi} (1 - r^2) r \, dr \, d\theta$.

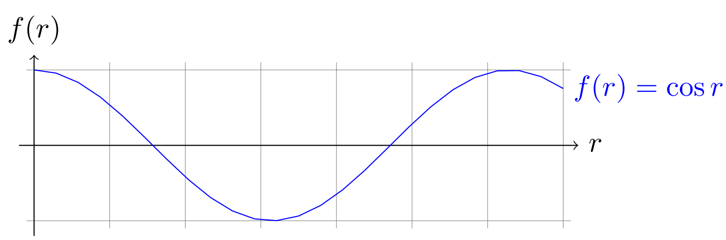


FIGURE 1.

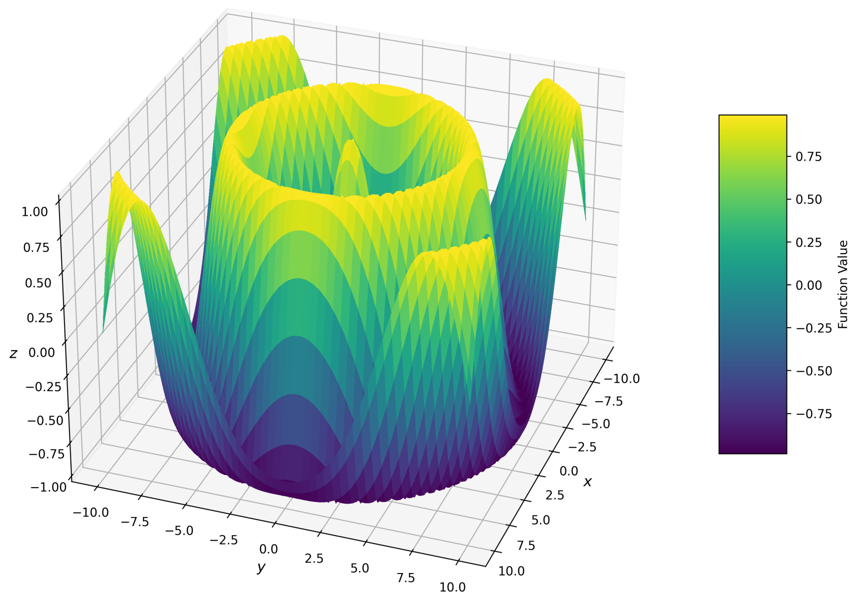


FIGURE 2. $z = \cos\sqrt{x^2 + y^2}$