MATH 233H EXAM I

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must show your work to receive credit for a solution; correct answers without justification are not sufficient to receive credit. Make sure you answer every part of every problem you do.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems from problems 2 and higher (problem 1 is automatically selected and you need not indicate it); any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

No electronic devices, such as phones, computers, tablets, smartwatches, or calculators may be used during the exam.

Date: Wednesday, 15 October 2025.

(1) (20 points) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.

(a) (4 points) The tangent plane to the unit sphere $x^2 + y^2 + z^2 = 1$ at the point $(0, 0, 1)$ is given by the equation $z = 1$.

(b) (4 points) If $\mathbf{v}$ is a vector, then a unit vector in the same direction as $\mathbf{v}$ is given by $\mathbf{v}/|\mathbf{v}|$.

(c) (4 points) If $f = xyz$ then $\nabla f = \langle yz, xz, xy\rangle$.

(d) (4 points) The directional derivative of $f(x, y) = x^2 + y^2$ at $(2, 1)$ in the direction of $\mathbf{i} + \mathbf{j}$ is 6.

(e) (4 points) Let (a, b) be a critical point of the function $f(x, y)$. If the Hessian D vanishes at (a, b), then the graph of f cannot have a local maximum at (a, b).

(2) (20 points) Let $\mathbf{a} = \langle 1, 0, -1\rangle$, $\mathbf{b} = \langle 1, 1, 1\rangle$, and $\mathbf{c} = \langle 0, 2, 3\rangle$. Compute the following.

(a) $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

(b) The equation of the line through the origin in the direction of $\mathbf{a}$.

(c) The area of the triangle with vertices given by the tips of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(d) The volume of the parallelepiped with edges along the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(3) (20 points) Two lines are given by the equations $\mathbf{r}_1(t) = \langle 3t + 1, -t, 2t + 1\rangle$ and $\mathbf{r}_2(t) = \langle s - 1, -s, s + 1\rangle$.

(a) (10 points) Determine if the lines coincide, are parallel, or are skew.

(b) (10 points) If the lines coincide, then write an equation for a line that parallel to them. If the lines do not coincide, then determine the distance between them,

(4) (20 points) The two parts of this problem are unrelated.

(a) (10 points) Suppose $\mathbf{f}(t) = \langle t, t^2, 2t^3/3\rangle$. Find the length of $\mathbf{f}(t)$ for $0 \le t \le 3$.

(b) (10 points) Suppose the acceleration of a particle's motion $\mathbf{r}(t)$ for $t \ge 0$ is given by $\mathbf{a}(t) = \langle \sin t, t, 2t + 1\rangle$ and that the initial velocity is $\mathbf{v}(0) = \langle 1, 0, 0\rangle$ and the initial position $\mathbf{r}(0)$ is at the origin. Find the position function $\mathbf{r}(t)$.

(5) (20 points) The temperature at a point (x, y, z) is given by $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$. Here T is measured in °C and x, y, z are measured in meters. Let P be the point $(2, -1, 2)$.

(a) (6 points) Find the rate of change of T at the point P when one moves in the direction of the point $(3, 0, 3)$.

(b) (7 points) In what direction does the temperature increase the fastest at P?

(c) (7 points) Find the maximum rate of increase of T at P.

(6) (20 points) Suppose $f(x, y) = x^2y + 12x^2 - 9y$.

(a) (8 points) Find and classify the critical points of f.

(b) (12 points) Find the absolute maximum and minimum of f on the closed rectangle with vertices $(0, 0)$, $(1, 0)$, $(0, 2)$, $(1, 2)$.

(7) (20 points) A plane P has the equation $Ax + y + 2z = 5$. Find all values of A that make the distance from P to the origin equal to 1.

(8) (20 points) Suppose that $z = f(x, y)$ and that $x = r\cos\theta$, $y = r\sin\theta$.

(a) (10 points) Use the chain rule to compute $\partial z/\partial r$ and $\partial z/\partial\theta$. (Hint: your answer may involve $\partial z/\partial x$ and $\partial z/\partial y$.)

(b) (10 points) Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$