Check for mistakes

233H Fall 2025 Exam 1 Answers

(1) (a) True (b) False (needs $\vec{x} \neq \vec{0}$).
(c) True (d) False ($\hat{\imath}+\hat{\jmath}$ not a unit vector)
(e) False (take $(a,b)=(0,0)$ and $f=-x^4-y^4$.)

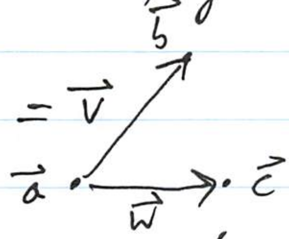
(2) (a) $\vec{a}\cdot\vec{b} = 1\cdot 1 + 0\cdot 1 + (-1)\cdot 1 = \boxed{0}$

$\vec{a}\times\vec{b} = \det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \boxed{\langle 1,-2,1\rangle}$

(b) $x=t,\ y=0,\ z=-t$ where $t\in\mathbb{R}$

(c)

$\langle 0,1,2\rangle = \vec{v}$



$\vec{w} = \langle -1,2,4\rangle$

need $\frac{1}{2}|\vec{v}\times\vec{w}|$ and

$\vec{v}\times\vec{w} = \det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 2 \\ -1 & 2 & 4 \end{pmatrix}$

$= \langle 0,-2,1\rangle$

so $\frac{1}{2}|\langle 0,-2,1\rangle|$

$= \boxed{\sqrt{5}/2}$

(d) The vectors go along the edges of the para. so we want $|\vec{a}\times\vec{b}\cdot\vec{c}|$ or $|\langle 1,-2,1\rangle\cdot\langle 0,2,3\rangle| = |-1| = \boxed{1}$

(3) (a) Direction vectors are $\vec{v}_1 = \langle 3,-1,2\rangle$ and $\vec{v}_2 = \langle 1,-1,1\rangle$. Not multiples so can't coincide or be parallel. Thus they either intersect or are skew. Thus $\boxed{\text{skew}}$ since that's the only option.

③ ⓑ common $\perp$ is given by $\vec{v}_1 \times \vec{v}_2$

$= \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 1 & -1 & 1 \end{pmatrix} = \langle 1, -1, -2 \rangle =: \vec{n}$

Need unit vector and $\hat{n} = \frac{1}{|\vec{n}|} \vec{n}$

$= \frac{1}{\sqrt{6}} \vec{n} = \frac{1}{\sqrt{6}} \langle 1, -1, -2 \rangle.$

vector $\vec{z}$ going between is given by $\langle 1, 0, 1 \rangle - \langle -1, 0, 1 \rangle$ (put $t = s = 0$ to get these points) $= \langle 2, 0, 0 \rangle$.

so the distance is $|\vec{z} \cdot \hat{n}| = \boxed{\frac{2}{\sqrt{6}}}$

④ ⓐ We want $\int_0^3 |\vec{f}\,'| \, dt$.

$\vec{f}\,' = \langle 1, 2t, 2t^2 \rangle$ and $|\vec{f}\,'| = \sqrt{1 + 4t^2 + 4t^4}$.

We see $1 + 4t^2 + 4t^4 = (1 + 2t^2)^2$ and since $1 + 2t^2 > 0$ we get $|\vec{f}\,'| = 1 + 2t^2$.

so $\int_0^3 1 + 2t^2 \, dt = t + \frac{2}{3} t^3 \Big|_0^3 = 3 + 2 \cdot 9$
$= \boxed{21}$

ⓑ $\vec{a} = \langle \sin t, t, 2t + 1 \rangle$
$\vec{v}(0) = \langle 1, 0, 0 \rangle$, $\vec{r}(0) = \langle 0, 0, 0 \rangle$
$\vec{v} = \int \vec{a} \, dt = \langle -\cos t, t^2/2, t^2 + t \rangle + \vec{C}$.

at $t = 0$ get $\langle -1, 0, 0 \rangle + \vec{C} = \langle 1, 0, 0 \rangle$
so $\vec{C} = \langle 2, 0, 0 \rangle$ and

4) b) (cont'd) $\vec{v}(t) = \langle 2 - \cos t, \ t^2/2, \ t^2 + t \rangle$.

$\vec{r} = \int \vec{v}\, dt = \langle 2t - \sin t, \ t^3/6, \ t^3/3 + t^2/2 \rangle + \vec{C}'$.

at $t = 0$ get $\langle 0,0,0 \rangle + \vec{C}' = \langle 0,0,0 \rangle$

so $\vec{C}' = \vec{0}$. Thus

$$\boxed{\vec{r}(t) = \langle 2t - \sin t, \ t^3/6, \ t^3/3 + t^2/2 \rangle}$$

5) a) $T = 200 \exp(-x^2 - 3y^2 - 9z^2)$

$\nabla T = -400 \exp(\cdots) \cdot \langle x, 3y, 9z \rangle$.

@ $(2,-1,2)$ get $-400 e^{-43} \langle 2, -3, 18 \rangle$

direction vector is $\langle 3,0,3 \rangle - \langle 2,-1,2 \rangle$

$= \langle 1,1,1 \rangle =: \vec{u}$, so $\hat{u} = \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle$.

and we want $D_{\vec{u}} T \big|_{\langle 2,-1,2 \rangle} =$

$-400 e^{-43} \langle 2,-3,18 \rangle \cdot \langle 1,1,1 \rangle / \sqrt{3}$

$= -17 \cdot 400 e^{-43} / \sqrt{3} = \boxed{-\frac{6800}{\sqrt{3}} e^{-43}} \ \frac{{}^\circ C}{m}$

(Note: $e^{-43} \approx 10^{-19}$ so this is an absurd number)

b) This is the direction of the gradient at that point, which is the same as the direction of ~~[illegible]~~ $\langle -2, 3, -18 \rangle =: \vec{z}$ (since $400 e^{-43} > 0$). $|\vec{z}| = \sqrt{4 + 9 + 324}$ $= \sqrt{337}$ so we get $\boxed{\frac{1}{\sqrt{337}} \langle -2, 3, -18 \rangle}$

5 (c) This is $|\nabla T|$ at $(2,-1,2)$, which is

$$\left| -400e^{-43} \langle 2,-3,18 \rangle \right|$$

$$= \boxed{400e^{-43}\sqrt{337}\ {}^{\circ}C/m}$$

6 (a) $f = x^2y + 12x^2 - 9y$

$\nabla f = \langle 2xy + 24x, x^2 - 9 \rangle \overset{?}{=} \vec{0}$

$\Longleftrightarrow$ $x^2 = 9$ so $x = \pm 3$ and

$2x(y+12) = 0 \Rightarrow y = -12.$

So 2 crit pts: $(-3,-12)$, $(3,-12)$.

Hessian is $D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \det\begin{pmatrix} 2y+24 & 2x \\ 2x & 0 \end{pmatrix}$

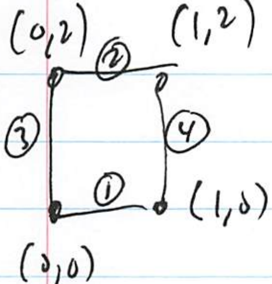
$= -4x^2.$

So $D < 0$ at ~~all~~ both crit pts and they are both saddle pts

$\Rightarrow$ Saddle pts @ $(-3,-12)$, $(3,-12)$

(b) Both crit pts from (a) are outside the domain. The four vertices are $(0,0)$, $(1,0)$, $(0,2)$, $(1,2)$ and f equals 0, 12, -18, -4 there

we check the sides of the rectangle



① $y=0 \Rightarrow f = 12x^2,\ f' = 24x \Rightarrow x=0$

(0,0) already in list

② $y=2 \Rightarrow f = 2x^2+12x^2-18$

$f' = 28x \Rightarrow x=0$

(0,2) already in list

③ $x=0 \Rightarrow f = -9y,\ f' = -9$

nothing from here

④ $x=1 \Rightarrow f = y+12-9y$

$= -8y+12$

$f' = -8 \Rightarrow$ nothing from here.

Looking at our list, we see

$\boxed{\text{MAX VAL is } 12,\ \text{MIN VAL is } -18}$

⑦ $P: Ax+y+2z=5$. normal vector is $\vec{n} = \langle A,1,2\rangle$, and $|\vec{n}| = \sqrt{A^2+5}$ so $\hat{n}$ is $\frac{1}{\sqrt{A^2+5}}\langle A,1,2\rangle$. Pt on plane is

$(0,5,0)$ so vector to origin is $\vec{z} = \langle 0,5,0\rangle$ and we want $|\vec{z}\cdot\hat{n}| = 1$

so $\left|5/\sqrt{A^2+5}\right| = 1$ or $5 = \sqrt{A^2+5}$

$25 = A^2+5$

$A^2 = 20$

and $\boxed{A = \pm 2\sqrt{5}}$

(8) (a) $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

By chain rule, $z_r = z_x x_r + z_y y_r$

$z_\theta = z_x x_\theta + z_y y_\theta$.

Now $x_r = \cos\theta$, $y_r = \sin\theta$

$x_\theta = -r\sin\theta$, $y_\theta = r\cos\theta$

and thus

$$\boxed{\begin{aligned} z_r &= z_x \cos\theta + z_y \sin\theta \\ z_\theta &= z_x(-r\sin\theta) + z_y(r\cos\theta). \end{aligned}}$$

(b) Compute the Right side of the identity:

$$z_r^2 + \frac{1}{r^2} z_\theta^2 = (z_x\cos\theta + z_y\sin\theta)^2 + \frac{1}{r^2}(z_x(-r\sin\theta) + z_y(r\cos\theta))^2$$

$$= z_x^2\cos^2\theta + \cancel{2z_xz_y\cos\theta\sin\theta} + z_y\sin^2\theta + \frac{r^2}{r^2}\left(z_x^2\sin^2\theta - \cancel{2z_xz_y\cos\theta\sin\theta} + z_y\cos^2\theta\right)$$

$$= z_x^2 + z_y^2 \quad ✓$$