MATH 233H EXAM II

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must show your work to receive credit for a solution; correct answers without justification are not sufficient to receive credit. Make sure you answer every part of every problem you do.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems from problems 2 and higher (problem 1 is automatically selected and you need not indicate it); any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

No electronic devices, such as phones, computers, tablets, smartwatches, or calculators may be used during the exam.

Date: Wednesday, 5 November 2025.

(1) (20 points) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.

(a) (4 points) The element of area dA in polar coordinates is $r\,dr\,d\theta$.

(b) (4 points) The graph of $r = 1$ in polar coordinates is a circle.

(c) (4 points) $\int_0^1 \int_0^2 x\,dx\,dy = \int_0^1 \int_0^2 y\,dy\,dx$.

(d) (4 points) The element of volume dV in spherical coordinates is $dV = \rho \sin\phi\,d\rho\,d\theta\,d\phi$.

(e) (4 points) Let R be a region in the plane and $f(x, y)$ a function. Then $\iint_R f(x, y)\,dA$ computes the area under the graph of f and over the region R.

(2) (20 points) Find the absolute maximum and minimum values of the function $f(x, y, z) = 2x + 6y + 10z$ on the graph of $x^2 + y^2 + z^2 = 35$.

(3) (20 points) Compute the following.

(a) (6 points)

$$\int_1^2 \int_y^2 xy\,dx\,dy.$$

(b) (7 points)

$$\iint_R \frac{2y}{x^2+1}\,dA,$$

where R is the region bounded by $0 \le y \le 1$, $y^2 \le x \le 1$.

(c) (7 points) The volume of the solid in the first octant under the plane $x + 2y - z = 0$ and over the bounded region in the xy plane bounded by $y = x^2$ and $y = x^3$.

(4) (20 points) Let $f(\theta) = \cos 3\theta$ (polar coordinates).

(a) (6 points) Plot the graph of $r = f(\theta)$.

(b) (7 points) Set up an integral that computes the area enclosed by the graph of f, but do not evaluate it.

(c) (7 points) Evaluate the integral to find the area.

(5) (20 points) Find the surface area of the portion of the graph $z = 2 - x^2 - y^2$ that lies above the xy-plane.

(6) (20 points) Compute the following integrals.

(a) (10 points)

$$\int_0^1 \int_0^z \int_0^{x+z} 6xz\,dy\,dx\,dz.$$

(b) (10 points)

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)\,dz\,dy\,dx.$$

(7) (20 points) A 3D solid E is bounded by the plane $z = 4$ and the graph of $z = x^2 + y^2$. Its density is given by $\rho(x, y, z) = 16 - z^2$.

(a) (8 points) Find the mass of E.

(b) (12 points) Find the center of mass of E.

(8) (20 points) Find the mass and center of mass of a solid hemisphere of radius a if the density at any point is proportional to the distance to the base.

MATH 233H FALL 2025 EXAM II
ANSWERS

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① (a) True (b) True (c) True (both equal 2)
(d) False (needs ρ^2, not ρ)
(e) False. Many things wrong (double integrals like this compute signed volume)

② $f = 2x + 6y + 10z$, $g = x^2 + y^2 + z^2 - 35 = 0$

$\nabla f = \lambda \nabla g \Rightarrow 2 = 2\lambda x,\ 6 = 2\lambda y,\ 10 = 2\lambda z$

since $\lambda \neq 0$ we get $x = \frac{1}{\lambda}$, $y = \frac{3}{\lambda}$, $z = \frac{5}{\lambda}$

$g = 0 \Rightarrow \frac{1}{\lambda^2} + \frac{9}{\lambda^2} + \frac{25}{\lambda^2} = 35$ so $\lambda = \pm 1$

$\Rightarrow$ candidates for Max/Min are
$(x, y, z) = (1, 3, 5)$ and $(-1, -3, -5)$.

Plug into f, get $\boxed{\begin{array}{l} 70 \text{ for 1st MAX} \\ -70 \text{ for 2nd MIN} \end{array}}$

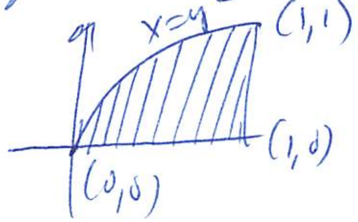
③ (a) $\int_1^2 \int_y^2 xy \, dx \, dy = \frac{1}{2}\int_1^2 x^2 y \Big|_y^2 dy = \frac{1}{2}\int_1^2 4y - y^3 \, dy$

$= \frac{1}{2}\left(2y^2 - \frac{1}{4}y^4\right)\Big|_1^2 = \frac{1}{2}$

$= y^2 - \frac{y^4}{8}\Big|_1^2 = \boxed{\frac{9}{8}}$

3b) Change the order. The region is $0 \le y \le 1$, $y^2 \le x \le 1$

or



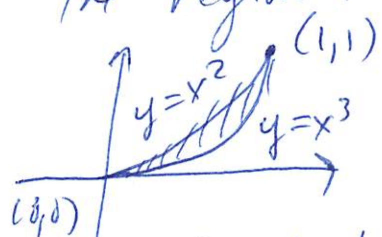
so $0 \le x \le 1$, $0 \le y \le \sqrt{x}$

and then $\displaystyle\int_0^1 \int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx = \int_0^1 \frac{(y^2)\Big|_0^{\sqrt{x}}}{x^2+1}\,dx$

$\displaystyle = \int_0^1 \frac{x}{x^2+1}\,dx \qquad u = x^2+1, \quad \tfrac{1}{2}du = x\,dx \qquad @\ x=0\ u=1,\ x=1\ u=2$

so $\displaystyle \frac{1}{2}\int_1^2 \frac{du}{u} = \frac{1}{2}\ln u\Big|_1^2 = \boxed{\frac{\ln 2}{2}}$

3c) The region in the plane looks like



(beware x^3 is under x^2)

so $0 \le x \le 1$, $x^3 \le y \le x^2$

The height is given by $z = x+2y$ and this is ≥ 0 if $x, y \ge 0$ so always over the xy plane.

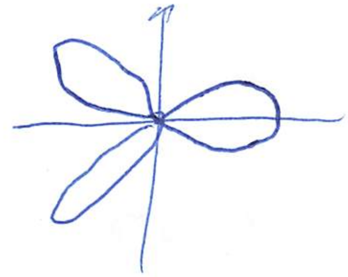
Therefore we want

$\displaystyle \int_0^1 \int_{x^3}^{x^2} (x+2y)\,dy\,dx = \int_0^1 xy + y^2\Big|_{x^3}^{x^2}\,dx$

$\displaystyle = \int_0^1 x^3 + \cancel{x^4} - \cancel{x^4} - x^6\,dx = \frac{1}{4}x^4 - \frac{1}{7}x^7\Big|_0^1$

$= \boxed{3/28}$

4 (a) $r = \cos 3\theta$



around $\theta = 0$, the value of r gets to 0 when $\theta = -\frac{\pi}{6}, \frac{\pi}{6}$ (since $\cos(\pm\frac{\pi}{2}) = 0$)

(b) we can do $3 \times$ 1 leaf or $6 \times \frac{1}{2}$ leaf

$(-\frac{\pi}{6} \le \theta \le \frac{\pi}{6})$ $\quad$ $(0 \le \theta \le \frac{\pi}{6})$

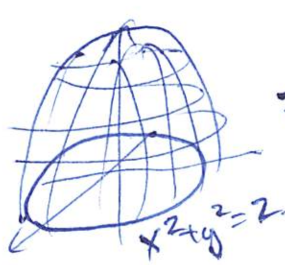
so $$A = 6\int_0^{\frac{\pi}{6}} \int_0^{\cos 3\theta} dA = \boxed{6\int_0^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r \, dr \, d\theta}$$

(c) $$3\int_0^{\frac{\pi}{6}} r^2 \Big|_0^{\cos 3\theta} d\theta = 3\int_0^{\frac{\pi}{6}} \cos^2 3\theta \, d\theta$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{6}} 1 + \cos 6\theta \, d\theta = \frac{3}{2}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_0^{\frac{\pi}{6}}$$

$$= \boxed{\frac{\pi}{4}}$$

5



$z = 2 - x^2 - y^2$
$= f$

Integrand: $\sqrt{1 + f_x^2 + f_y^2}$
$= \sqrt{1 + 4x^2 + 4y^2}$

Base region $x^2 + y^2 \le 2$

$\Longrightarrow$ better to use polar! base is $0 \le \theta \le 2\pi$
$0 \le r \le \sqrt{2}$

Integrand is $\sqrt{1 + 4r^2}$

(5) (continued.)

$$\text{so} \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1+4r^2}\, r\,dr\,d\theta$$

$u = 1+4r^2$

$\frac{1}{8}du = r\,dr$

@ $x=0$ $u=1$

@ $x=\sqrt{2}$ $u=9$

$$= \frac{1}{8}\int_0^{2\pi}\int_1^9 u^{1/2}\,du\,d\theta$$

$$= \frac{2}{3}\cdot\frac{1}{8}\int_0^{2\pi} u^{3/2}\Big|_1^9\,d\theta$$

$$= \frac{1}{12}(27-1)\int_0^{2\pi} d\theta = \boxed{\frac{13\pi}{3}}$$

(6) (a) $$6\int_0^1\int_0^z\int_0^{x+z} xz\,dy\,dx\,dz$$

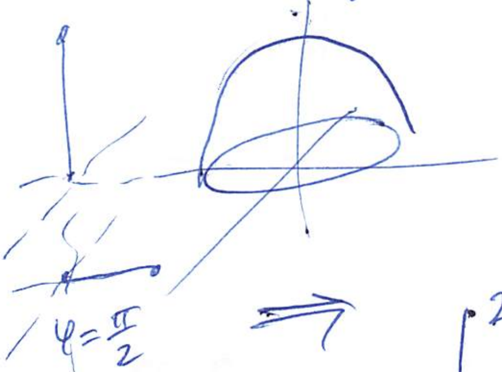
$$= 6\int_0^1\int_0^z x^2z + xz^2\,dx\,dz$$

$$= 6\int_0^1 \frac{1}{3}x^3z + \frac{1}{2}x^2z^2\Big|_0^z\,dz$$

$$= 6\cdot\frac{5}{6}\int_0^1 z^4\,dz = z^5\Big|_0^1 = 1$$

(6)(b) $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \underbrace{(x^2+y^2+z^2)}_{\rho^2}\,dz\,dy\,dx$$

unit circular disk in xy plane; upper $\frac{1}{2}$ of solid unit sphere $\Rightarrow$ spherical is better



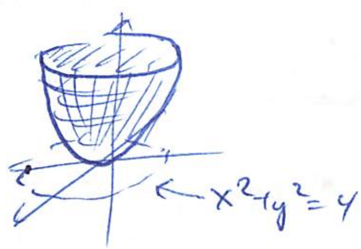
$0\le\theta\le 2\pi$ $0\le\varphi\le\pi/2$ (not π)
$0\le\rho\le 1$ $dV=\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$$\Rightarrow \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^4\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=\frac{1}{5}\int_0^{2\pi}\int_0^{\pi/2}\sin\varphi\,d\varphi\,d\theta$$

$$=\frac{1}{5}\int_0^{2\pi}\underbrace{(-\cos\varphi)\Big|_0^{\frac{\pi}{2}}}_{1}\,d\theta = \frac{1}{5}\int_0^{2\pi}d\theta = \boxed{\frac{2\pi}{5}}$$

(7) $z=4$, $z=x^2+y^2 \Rightarrow$
density $=16-z^2$.
Shape and density invariant under rotation about z-axis $\Rightarrow \bar{x}=\bar{y}=0$. so only need $\bar{z}$.



Base is disk $x^2+y^2\le 2$, density depends only on z
$\Rightarrow$ cylindrical is best. The 3D solid is given by
$0\le\theta\le 2\pi$, $0\le r\le 2$
$r^2\le z\le 4$.

7 (continued)

a $M = \int_0^2 \int_{r^2}^4 \int_0^{2\pi} (16 - z^2)\, r \, d\theta \, dz \, dr$

$= 2\pi \int_0^2 r\left(16z - \frac{z^3}{3}\right)\Big|_{r^2}^{4} dr$

$= 2\pi \int_0^2 r\left[\left(64 - \frac{64}{3}\right) - \left(16r^2 - \frac{r^6}{3}\right)\right] dr$

$= 2\pi \int_0^2 \left(\frac{2}{3}\cdot 64 r - 16r^3 + \frac{r^7}{3}\right) dr$

$= 2\pi \left(\frac{64}{3} r^2 - 4r^4 + \frac{r^8}{24}\right)\Big|_0^2$

$= 2\pi \left(\frac{256}{3} - \frac{256}{4} + \frac{256}{24}\right) = 512\pi\left(\frac{1}{3} - \frac{1}{4} + \frac{1}{24}\right)$

$= \boxed{64\pi}$

b only need the moment $\int_0^2 \int_{r^2}^4 \int_0^{2\pi} (16z - z^3)\, r \, d\theta \, dz \, dr$

$= 2\pi \int_0^2 r\left(8z^2 - \frac{z^4}{4}\right)\Big|_{r^2}^{4} dr$

$= 2\pi \int_0^2 r\left(128 - 64 - 8r^4 + \frac{r^8}{4}\right) dr$

$= 2\pi \int_0^2 \left(64r - 8r^5 + \frac{r^9}{4}\right) dr$

$= 2\pi \left(32r^2 - \frac{4}{3}r^6 + \frac{r^{10}}{40}\right)\Big|_0^2$

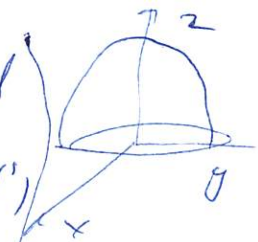
$= 2\pi \left(\frac{64\cdot 4}{2} - \frac{64\cdot 4}{3} + \frac{256}{10}\right)$

$= 2\pi\cdot 256\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}\right) = 2\pi\cdot 256\cdot\frac{4}{15}$

$\bar{z} = \dfrac{2\pi\cdot 256\cdot\frac{4}{15}}{64\pi} = \boxed{\dfrac{32}{15}}$

6

(8) density $= Kz$ for some const K. Since shape and density are invariant under rotation about z-axis, $\bar{x} = \bar{y} = 0$ and only need $\bar{z}$.



use spherical.

$0 \le \rho \le a$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \frac{\pi}{2}$

Mass M: need $dV = \rho^2 \sin\varphi \, d\varphi \, d\theta \, d\rho$,

$z = \rho \cos\varphi$

$$M = K\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{2}} \cos\varphi \sin\varphi \, d\varphi\right)\left(\int_0^a \rho^3 \, d\rho\right)$$

$$\underset{2\pi}{\shortparallel} \qquad \underset{\frac{1}{2}\sin^2\varphi \Big|_0^{\pi/2} = 1/2}{\shortparallel} \qquad \underset{\frac{a^4}{4}}{\shortparallel}$$

since no functions in limits we can just do each integral separately (just saves some writing)

so mass is $\boxed{K\pi a^4/4}$.

$$\bar{z} \text{ is then } \frac{1}{M}\iiint_E Kz \cdot z \, dV$$

$$= \frac{4}{\pi a^4}\int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\cos\varphi)^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \frac{4}{\pi a^4}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{2}} \cos^2\varphi \sin\varphi\right)\left(\int_0^a \rho^4 \, d\rho\right)$$

$$= \frac{4}{\pi a^4} \cdot 2\pi \cdot \left(-\frac{1}{3}\cos^3\varphi \Big|_0^{\pi/2}\right) \cdot \frac{a^5}{5}$$

$$= \frac{8a}{5} \cdot \frac{1}{3} = \boxed{\frac{8a}{15}} \Longrightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{8a}{15}\right)$$