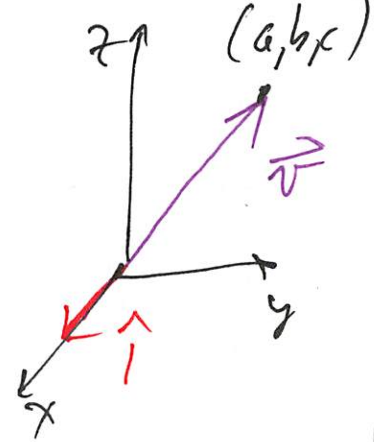
Last time: vectors.

$\vec{v} = \langle a,b,c \rangle$.

$\vec{w} = \langle d,e,f \rangle$



α scalar

$\alpha\vec{v} = \langle \alpha a, \alpha b, \alpha c \rangle$, $\vec{0} = \langle 0,0,0 \rangle$.

$\vec{v} + \vec{w} = \langle a+d, b+e, c+f \rangle$

$\vec{v} - \vec{w} = \langle a-d, b-e, c-f \rangle$

$|\vec{v}| =$ length $= \sqrt{a^2+b^2+c^2}$

$\hat{\ } \iff$ vector of length 1
$|\hat{v}| = 1$.

$|\vec{v}| = 0 \iff \vec{v} = \vec{0}$

if $\vec{v} \neq \vec{0}$, then $\frac{1}{|\vec{v}|}\vec{v}$ is a unit vector in direction of $\vec{v}$

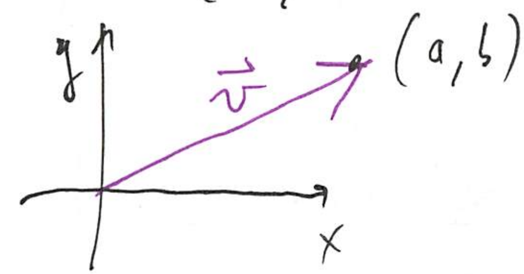
$\hat{i} = \langle 1,0,0 \rangle$
$\hat{j} = \langle 0,1,0 \rangle$
$\hat{k} = \langle 0,0,1 \rangle$

"standard basis vectors"

$$\vec{v} = \langle a,b,c \rangle = a\hat{i} + b\hat{j} + c\hat{k} = \langle a,0,0 \rangle + \langle 0,b,0 \rangle + \langle 0,0,c \rangle$$

2D vectors too:

$\vec{v} = \langle a,b \rangle$



Have 3 different kinds of multiplication. one is scalar mult. 2 others that take 2 vectors as input.

	Input	Output
dot products	2 vectors	scalar
cross product	2 vectors	vector.

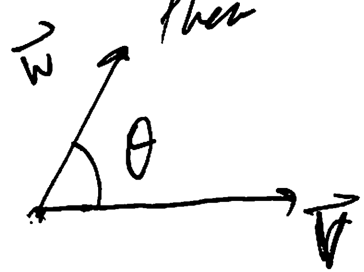
<u>Dot product.</u>

$$\vec{V}, \vec{W} \Longrightarrow \vec{V} \cdot \vec{W}$$

<u>Def</u> $\vec{V} \cdot \vec{W}$ is the scalar

$$|\vec{V}||\vec{W}| \cos \theta$$

where θ is angle between them



if $\vec{v} = \langle a, b, c \rangle$, ②
$\vec{w} = \langle d, e, f \rangle$, then

$$\vec{v} \cdot \vec{w} = ad + be + cf$$

Fact: same number!

e.g. $\vec{v} = \langle 1, -1, 0 \rangle$
$\vec{w} = \langle 0, 1, -1 \rangle$

$$\vec{v} \cdot \vec{w} = 1 \cdot 0 + (-1) \cdot 1 + 0 \cdot (-1) = -1$$

<u>Properties</u>

$$\vec{V} \cdot \vec{W} = \vec{W} \cdot \vec{V}$$

$$\vec{V} \cdot (\vec{u} + \vec{w}) = \vec{V} \cdot \vec{u} + \vec{V} \cdot \vec{w}$$

$$(\alpha \vec{V}) \cdot \vec{W} = \alpha (\vec{V} \cdot \vec{W})$$

we can use the dot product to understand the angle θ

e.g. 2 vectors are $\perp \iff$

$\vec{v} \cdot \vec{w} = 0$

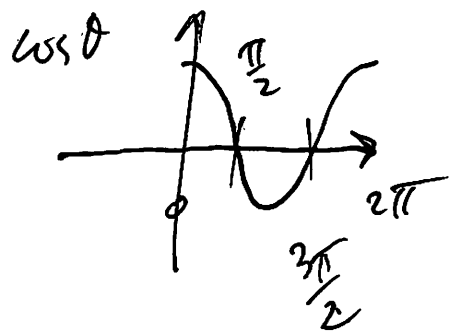
(assume $\vec{v}, \vec{w} \neq \vec{0}$)

Why? $\underbrace{\vec{v} \cdot \vec{w}}_{0} = |\vec{v}||\vec{w}| \cos\theta$

$\Longrightarrow \cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\frac{\pi}{2} = 90^\circ$



e.g. $\langle 1,1,1\rangle \cdot \langle 1,0,-1\rangle = 0$

e.g. can find angle between $\vec{v}, \vec{w}$.

e.g. $\vec{v} = \langle 1,-1,0\rangle$

$\vec{w} = \langle 0,1,-1\rangle$

want θ.

$$\begin{cases} \vec{v} \cdot \vec{w} = -1 \\ \phantom{\vec{v} \cdot \vec{w}} = |\vec{v}||\vec{w}| \cos\theta \end{cases}$$

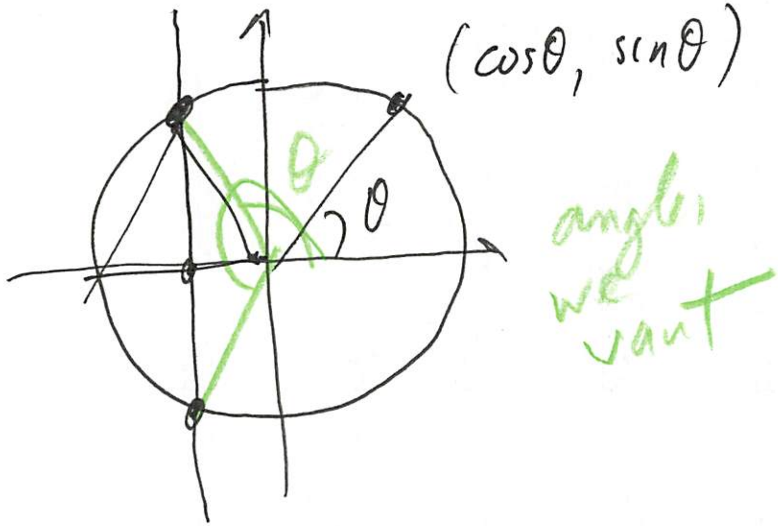
$|\vec{v}| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$|\vec{w}| = \sqrt{\cdots} = \sqrt{2}$

$\Longrightarrow -\frac{1}{\sqrt{2}\sqrt{2}} = \cos\theta$

or $\cos\theta = -\frac{1}{2}$.

use unit circle



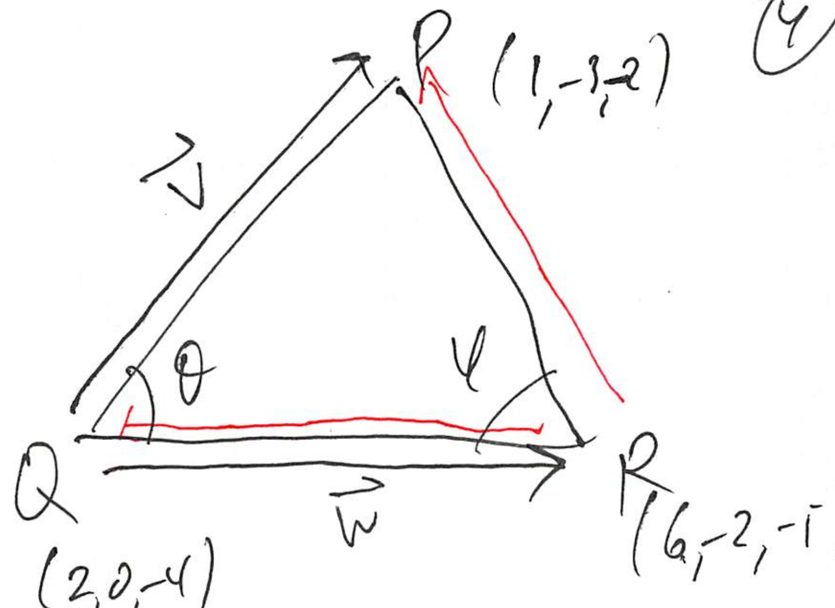
$$\Rightarrow \quad \theta = \frac{2\pi}{3}$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\,|\vec{w}|}$$

e.g. $P = (1,-3,-2)$
$Q = (2,0,-4)$
$R = (6,-2,-1)$

they determine a $\triangle$

is this a right $\triangle$?



need vectors along sides of $\triangle$.

$\vec{v} = \overrightarrow{QP}$

to get $\vec{v}$ take difference of coords.

$$\vec{v} = \langle 1-2, -3-0, -2--4 \rangle$$
$$= \langle -1, -3, 2 \rangle$$

$$\vec{w} = \overrightarrow{QR} = \langle 4, -2, -1 \rangle$$

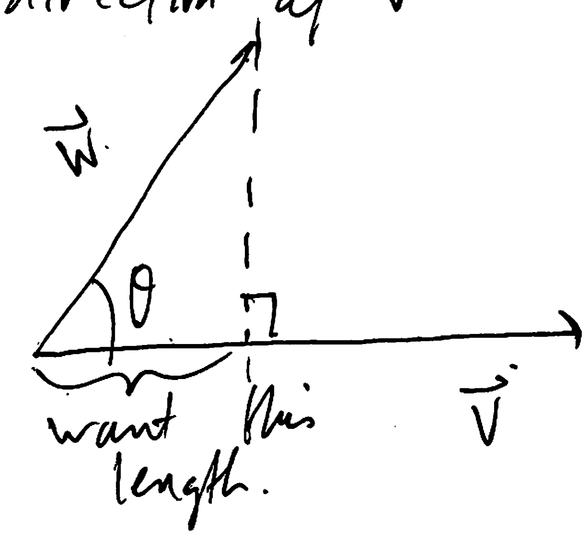
$\vec{v}\cdot\vec{w} = \langle -1,-3,2\rangle \cdot \langle 4,-2,-1\rangle$

$= -4+6-2 = 0$

$\Rightarrow \vec{v} \perp \vec{w} \Rightarrow \theta$ is right angle.

e.g. finding the component of $\vec{w}$ in direction of $\vec{v}$

This means finding how far $\vec{w}$ projects in the direction of $\vec{v}$.



desired distance is

$|\vec{w}|\cos\theta$

almost $\vec{v}\cdot\vec{w}$, missing $|\vec{v}|$

$\Rightarrow$ this component equals

$$\frac{1}{|\vec{v}|}\vec{v}\cdot\vec{w}$$

can also think of it as

$$\left(\frac{1}{|\vec{v}|}\vec{v}\right)\cdot\vec{w}$$

$$= \hat{v}\cdot\vec{w}$$

e.g. find component of $\vec{w} = \langle 1,-1,3\rangle$ in direction of $\langle 6,3,2\rangle = \vec{v}$

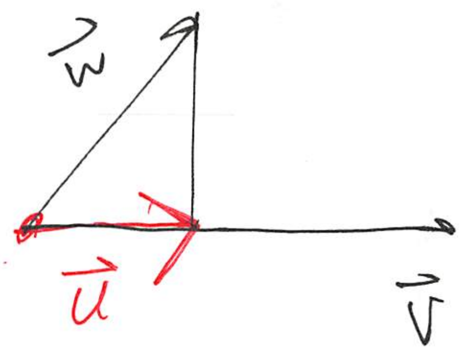
want $\vec{w} \cdot \hat{v}$.

$$|\vec{v}| = \sqrt{36+9+4} = \sqrt{49} = 7$$

$$\Rightarrow \hat{v} = \left\langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \right\rangle$$

$$\vec{w} \cdot \hat{v} = \frac{6}{7} - \frac{3}{7} + \frac{6}{7} = \boxed{\frac{9}{7}}$$

e.g. vector projection of of $\vec{w}$ along $\vec{v}$ means the the vector $\vec{u}$



has length $\hat{v} \cdot \vec{w}$.

direction is $\hat{v}$.

$$\Longrightarrow \vec{u} = (\hat{v} \cdot \vec{w})\, \hat{v}$$

previous example:

$$\hat{v} \cdot \vec{w} = \frac{9}{7}$$

$$\hat{v} = \left\langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \right\rangle$$

$$\Longrightarrow \vec{u} = \left\langle \frac{54}{49}, \frac{27}{49}, \frac{18}{49} \right\rangle$$

<u>Cross product</u> $\vec{v} \times \vec{w}$

input 2 vectors
output vector

only makes sense for vectors in 3D.

<u>geometric def.</u>

$\vec{v} \times \vec{w}$ is the vector with length = $|\vec{v}||\vec{w}| \sin\theta$

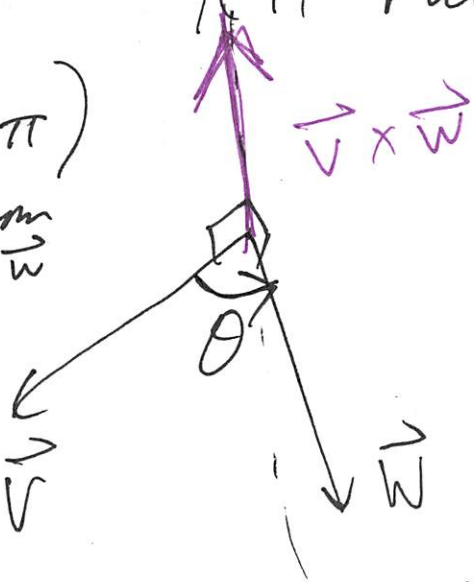
direction = $\perp$ to both $\vec{v}, \vec{w}$

and determined using

RH rule.

($\measuredangle\theta \le \pi$

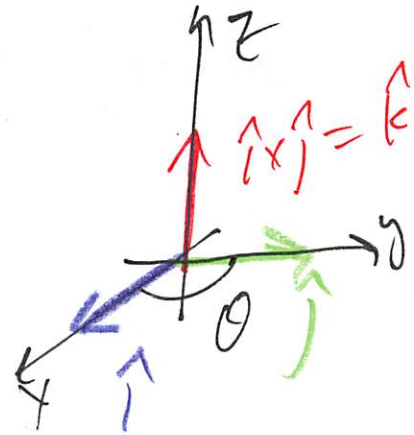
goes from $\vec{v}$ to $\vec{w}$)



push $\vec{w}$ into $\vec{w}$ along θ.

RH Thumb points in direction of $\vec{v} \times \vec{w}$

e.g. $\hat{i} \times \hat{j} = ?$



$\theta = \frac{\pi}{2}$

$\sin\frac{\pi}{2} = 1$

$|\hat{i}| = |\hat{j}| = 1$

$\hat{i} \times \hat{j} \perp$ to $\hat{i}, \hat{j}$

and has length 1.

$\Longrightarrow \hat{k}$ or $-\hat{k}$

$\hat{j} \times \hat{i} = \hat{k}$ or $-\hat{k}$

$= -\hat{k}$

$\Longrightarrow \boxed{\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}}$

NOT COMMUTATIVE

8

algebraic def

use determinants.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

2×2

need determinant of 3×3 matrix

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= aei + bfg + cdh - ceg - afh - bdi$$

can also compute in terms of 2×2 dets

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= +a \cdot \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

get same thing

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how use to compute $\vec{v} \times \vec{w}$.

$\vec{v} = \langle a, b, c \rangle$

$\vec{w} = \langle d, e, f \rangle$

① make 3x3 matrix

ⓐ 1st row $\hat{i}\ \hat{j}\ \hat{k}$

ⓑ 2nd row $\vec{v}$

ⓒ 3rd row $\vec{w}$

② take determinant, interpret result as a vector.

e.g. $\vec{v} = \langle 1, -1, 0 \rangle$

$\vec{w} = \langle 0, 1, -1 \rangle$

$\vec{v} \times \vec{w} = ?$

$$\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

can check $\vec{v} \times \vec{w} \perp \vec{v}, \vec{w}$

$$= +\hat{i} \det \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} - \hat{j} \det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \hat{k} \det \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= 1\hat{i} - (-1)\hat{j} + 1\hat{k}$$

$$= \langle 1, 1, 1 \rangle$$