Last time: dot, cross.

cross product $\vec{v} \times \vec{w}$

$|\vec{w} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$

direction determined by RH rule.

use 3×3 dets for computation

$\vec{v} = \langle a, b, c\rangle$, $\vec{w} = \langle d, e, f\rangle$

$$\vec{v} \times \vec{w} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$

$$= \hat{i} \det \begin{pmatrix} b & c \\ e & f \end{pmatrix} - \hat{j} \det \begin{pmatrix} a & c \\ d & f \end{pmatrix} + \hat{k} \det \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

$$= \langle bf - ce, cd - af, ae - bd \rangle$$

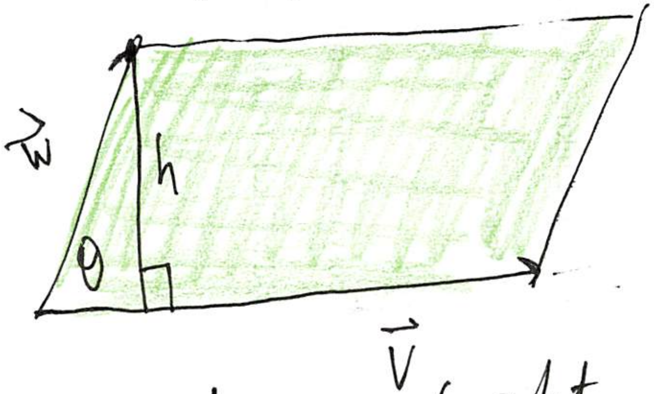
Properties

- not commutative
  $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
- $\vec{v} \times \vec{w} \perp$ both $\vec{v}, \vec{w}$.
  (check using dot prod)
- $\vec{v} \times (\vec{w} + \vec{u}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{u}$
- $(\alpha \vec{v}) \times \vec{w} = \alpha(\vec{v} \times \vec{w})$
  α scalar.

Applications

1. computing areas of parallelograms / triangles in 3D

Suppose have 2 vectors going along sides of para.



area = base · height

$= |\vec{v}| \cdot h$

$= |\vec{v}|\,|\vec{w}| \sin\theta$

$= |\vec{v} \times \vec{w}|$

e.g. find area of parall. with sides along $\vec{v} = \langle 1, 2, 3 \rangle$, $\vec{w} = \langle 1, 0, -1 \rangle$

need $|\vec{v} \times \vec{w}|$

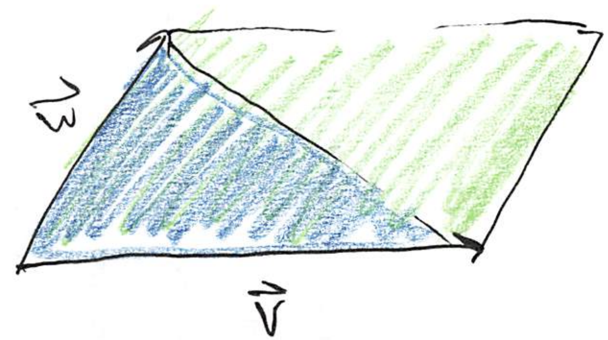
$$\vec{v} \times \vec{w} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \langle -2, 4, -2 \rangle$$

$\Rightarrow$ area $= \sqrt{2^2 + 4^2 + 2^2}$

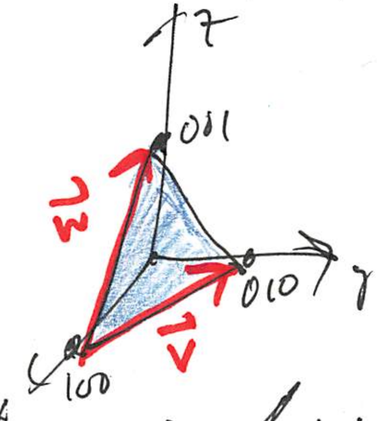
$= \sqrt{24} = 2\sqrt{6}$.

e.g. if $\vec{v}, \vec{w}$ make 2 sides of a Δ, we can find area.



area $\Delta = \frac{1}{2}$ area parall.

e.g. Find area of $\triangle$ with verts $P = (1,0,0)$, $Q = (0,1,0)$, $R = (0,0,1)$



$\vec{PQ} = \vec{v} = \langle -1,1,0\rangle$, $\vec{PR} = \vec{w} = \langle -1,0,1\rangle$

$$\vec{v}\times\vec{w} = \det\begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}$$

$$= \langle 1,1,1\rangle$$

$$\frac{1}{2}\left|\langle 1,1,1\rangle\right| = \boxed{\frac{1}{2}\sqrt{3}}.$$

(2) Triple scalar product. mixes dot, cross with 3 vectors

$\vec{v}, \vec{w}, \vec{u}$ ✓

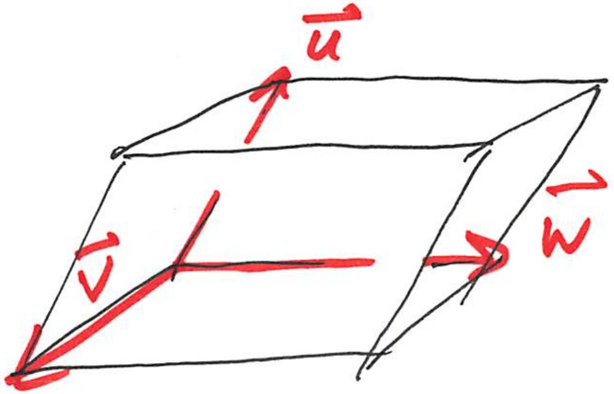
$$\vec{v}\times\vec{w}\cdot\vec{u} = (\vec{v}\times\vec{w})\cdot\vec{u}$$

$$\neq$$

$\vec{v}\times(\vec{w}\cdot\vec{u})$ makes no sense

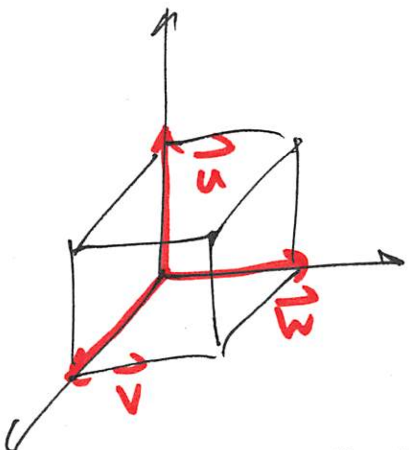
output is a scalar

The absolute value of this scalar is the volume of the parallelepiped (squashed box) spanned by $\vec{v}, \vec{w}, \vec{u}$.



Faces are parallelograms.

e.g. $\vec{v} = \hat{\imath}$, $\vec{w} = \hat{\jmath}$, $\vec{u} = \hat{k}$



unit cube.

volume is $|\vec{v} \times \vec{w} \cdot \vec{u}|$

(know ans = 1)

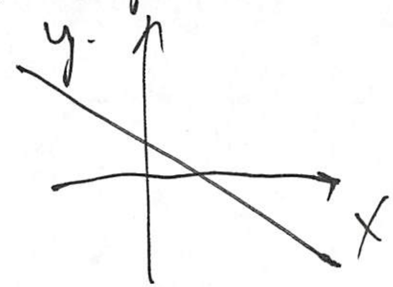
$$\vec{v} \times \vec{w} = \hat{\imath} \times \hat{\jmath} = \hat{k}$$

$$\vec{v} \times \vec{w} \cdot \vec{u} = \hat{k} \cdot \hat{k} = 1$$

Rem The output of TSP is unique up to a minus sign. So if want volume, just take absolute value.

Equations of lines and planes in 3D. vectors very helpful.

2D first.



eqn of line is

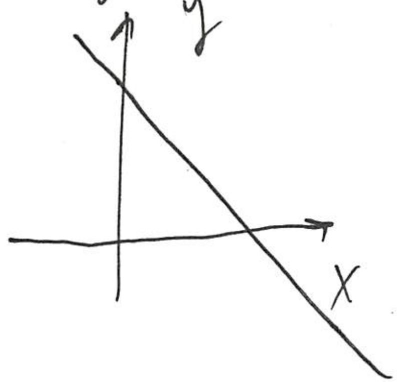
$$Ax + By = C$$

A, B, C constants.

we also have the parametric form of equation of a line

In parametric eqns, we introduce another variable t ("parameter") and give x, y as functions in t.
For a line, these are linear functions of t.

e.g.



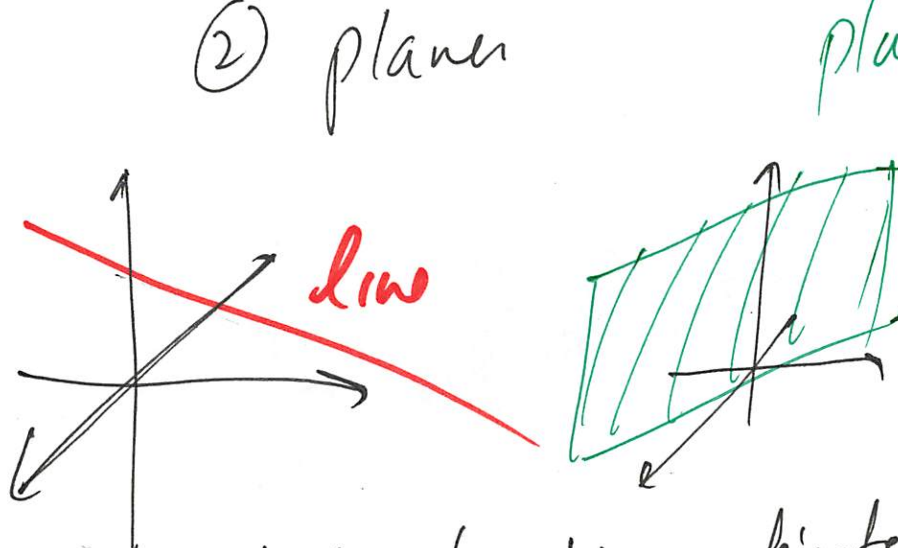
$x+y=1$
non-parametric form.

parametric form $\begin{cases} x = t \\ y = 1-t \end{cases}$

t	x	y
0	0	1
1	1	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

In 3D, have 2 kinds of flat geometric objects

(1) lines
(2) planes

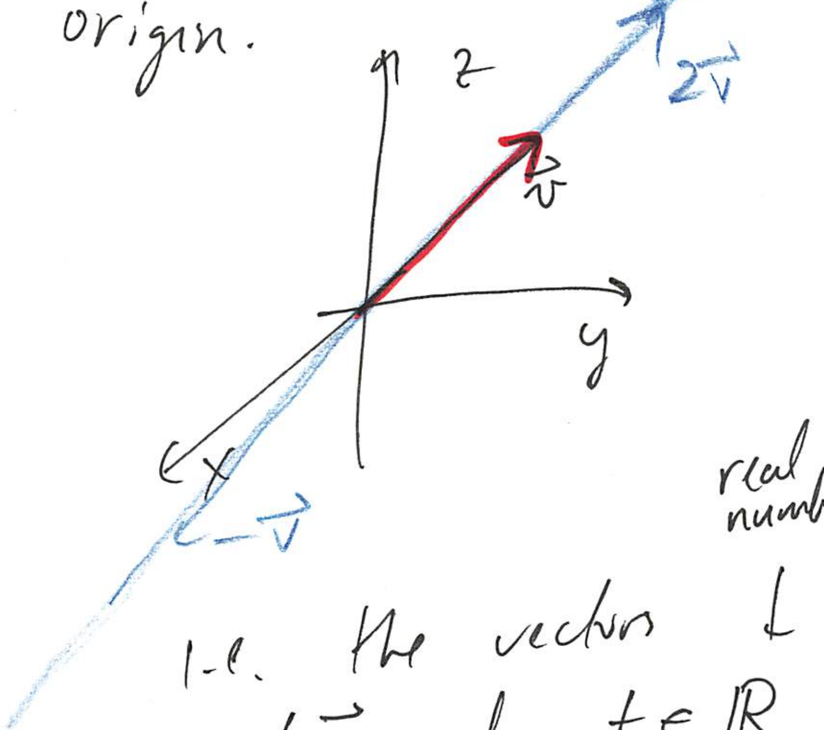


we will describe these objects using

(1) lines $\Longleftrightarrow$ parametric
(2) planes $\Longleftrightarrow$ non parametric

Lines. want to use vectors.

idea: fix vector $\vec{v}$. The points corresponding to all scalar multiples of $\vec{v}$ i.e on a line through the origin.



i.e. the vectors $t\vec{v}$ where $t \in \mathbb{R}$

"in"

real numbers

determine a line through the origin.

can use to get parametric eqns.

if $\vec{v} = \langle a, b, c \rangle$

then $t\vec{v} = \langle ta, tb, tc \rangle$

$$\Longleftrightarrow \begin{cases} x = ta \\ y = tb \\ z = tc. \end{cases}, \quad t \in \mathbb{R}$$

e.g. Find param. eqns for line through the origin and in the direction of $\vec{v} = \langle 1, 2, 3 \rangle$

ans: $\begin{cases} x = t \\ y = 2t \\ z = 3t \end{cases}, \quad t \in \mathbb{R}$

$\vec{v}$ is called the <u>direction vector</u> of the line.

To get param. eqns for a line not necc. through origin, we take an additional vector to give a point on the line.

direction vector $\quad \vec{v} = \langle a, b, c \rangle$

point on line $\quad \vec{X}_0 = \langle x_0, y_0, z_0 \rangle$

The eqns are then

$$t\vec{v} + \vec{X}_0$$

$$\Longleftrightarrow \boxed{\begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}, \quad t \in \mathbb{R}}$$

e.g. find eqns for line ℓ through $(1,1,1)$ and in direction of $\langle 2,3,7 \rangle$.

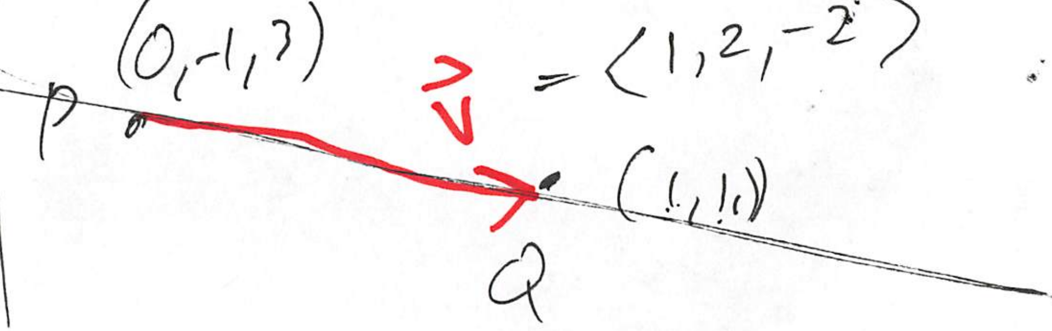
ans: $\begin{cases} x = \boxed{2}t + \boxed{1} \\ y = \boxed{3}t + \boxed{1} \\ z = \boxed{7}t + \boxed{1} \end{cases}, \quad t \in \mathbb{R}.$

direction vec. $\qquad$ pt on line

e.g. Find line through pts $P = (0,-1,3)$
$Q = (1,1,1)$

take $\vec{X}_0 = \langle 1,1,1 \rangle$.

$\vec{v} = \langle 1, 2, -2 \rangle$



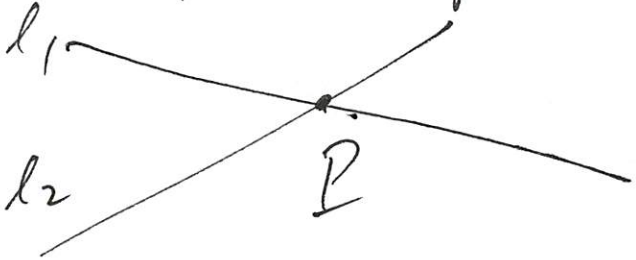
ans: $\begin{cases} x = t + 1 \\ y = 2t + 1 \\ z = -2t + 1 \end{cases} \quad t \in \mathbb{R}$

variant: $\begin{cases} x = t + 0 \\ y = 2t - 1 \\ z = -2t + 3 \end{cases} \quad t \in \mathbb{R}$

Same line. 2

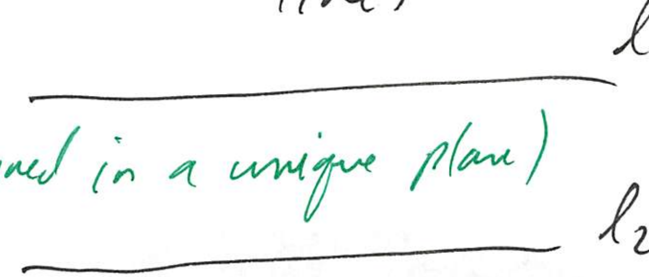
e.g. suppose have different l_1 l_2 lines in 3D. what can they look like?

① They can intersect in a unique pt P.

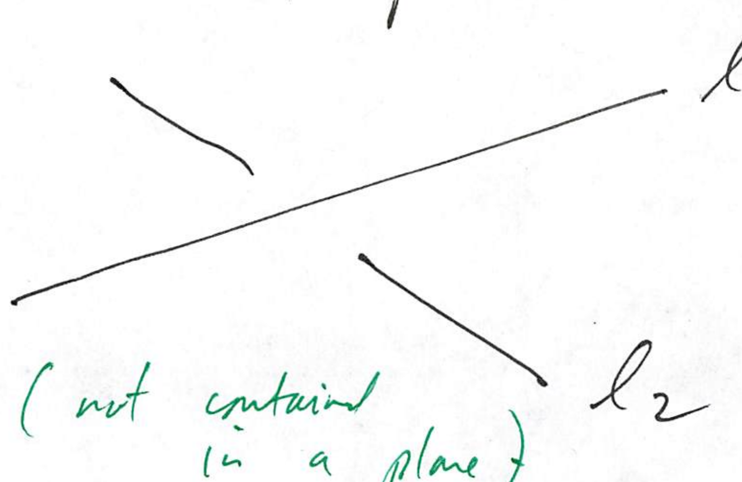


② They don't intersect

2a parallel lines



2b skew lines
don't intersect, but not parallel.



2 parallel lines have direction vectors that are scalar multiples of each other!
skew lines don't.

e.g. $l_1 = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$

$l_2 = \begin{cases} x = s \\ y = s + 2 \\ z = 2s + 1 \end{cases}$

direction vectors are
$\langle 1,1,1 \rangle$, $\langle 1,1,2 \rangle$
$\Longrightarrow$ not parallel.
claim skew.

to check, try to solve.

$\left.\begin{array}{l} s = t \\ s+2 = t \\ 2s+1 = t \end{array}\right\}$ $\begin{array}{c} s+2 = s \\ 2 = 0 \end{array}$

no soln.

$\Longrightarrow$ don't intersect.

e.g. $l_1 : \begin{cases} x = 3t+1 \\ y = t+2 \\ z = -2t+15 \end{cases}$

$l_2 : \begin{cases} x = 2s+1 \\ y = s \\ z = s+1 \end{cases}$

$\cap$? parallel? skew?
not parallel (look at dir. vects)

$y: s = t+2$ (*)

use in x: $3t+1 = 2(t+2)+1$

$\Rightarrow 3t+1 = 2t+4+1$

$t = \cancel{3}\ 4$

(*) $\Rightarrow s = \cancel{5}\ \cancel{7}\ 6$

need to check z eqn.

$l_1: -2\cdot\cancel{4}+15 = \cancel{8}\ 7$

$l_2: \cancel{6}+1 = \cancel{6}\ 7$

~~not~~ same!

~~don't~~ intersect!!!

$\Rightarrow$ ~~Skew~~.