Last time: Lines / Planes.

<u>Lines</u> $\vec{v}$ direction vector $\langle a,b,c\rangle$

$\vec{x}_0$ point on line $\langle x_0,y_0,z_0\rangle$

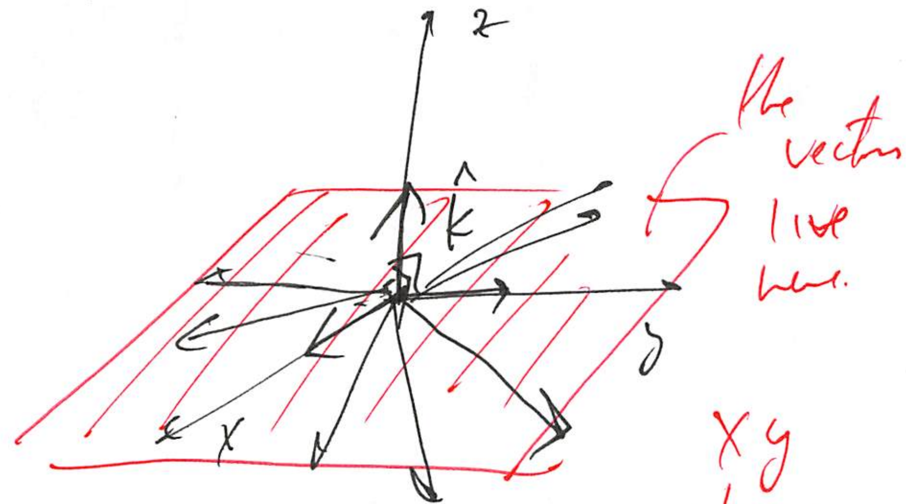
$\Rightarrow$ get parametric eqns for line.

$t\vec{v}+\vec{x}_0$

$$\Longleftrightarrow \begin{cases} x = at+x_0 \\ y = bt+y_0 \\ z = ct+z_0 \end{cases} \quad t\in\mathbb{R}.$$

<u>Planes</u> idea: the collection of vectors $\perp$ to a given vector $\vec{n}$ form a plane.

e.g. $\vec{n}=\hat{k}$ ①



To get eqn, suppose $\vec{n}=\langle A,B,C\rangle$, let $\vec{x}$ be a vector of variable $\vec{x}=\langle x,y,z\rangle$

$\vec{n}\perp\vec{x} \Longleftrightarrow \vec{n}\cdot\vec{x}=0$

$$\Longrightarrow \boxed{Ax+By+Cz=0}$$

eqn of plane through the origin.

e.g. $\vec{n} = \hat{k} = \langle 0,0,1 \rangle$
$\vec{x} = \langle x,y,z \rangle$

$\vec{n} \cdot \vec{x} = 0 \iff z = 0$

gives xy plane

$\vec{n}$ is called the normal vector for the plane. ⚠ means $\perp$

General plane: need normal vector $\vec{n}$ + a point on the plane. Eqn has form

$$\boxed{Ax + By + Cz = D}$$

$\vec{n} = \langle A,B,C \rangle$ as before.
find D by plugging in the point.

e.g. find plane $\perp$ to $\langle 1,2,3 \rangle$ and containing $(1,1,1)$.

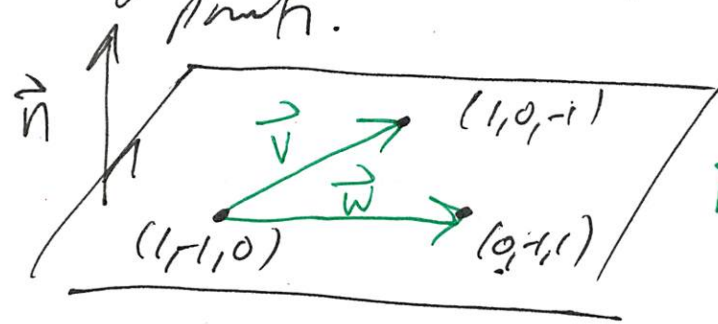
$$\overset{\vec{n}}{1x + 2y + 3z} = D$$

now plug in: $1+2+3 = D = 6$

ans: $\boxed{x + 2y + 3z = 6.}$

e.g. 3 pts, not collinear, determine a unique plane.

e.g. $(1,-1,0), (0,-1,1), (1,0,-1)$
find plane containing the points.



use cross product

$\vec{n} = \vec{v} \times \vec{w}$.

$\vec{v} = \langle 0, 1, -1 \rangle \quad \vec{w} = \langle -1, 0, 1 \rangle$

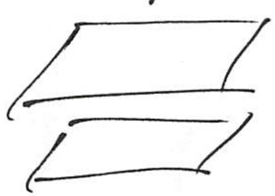
$$\vec{v} \times \vec{w} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \langle 1, 1, 1 \rangle = \vec{n} \quad \checkmark$$

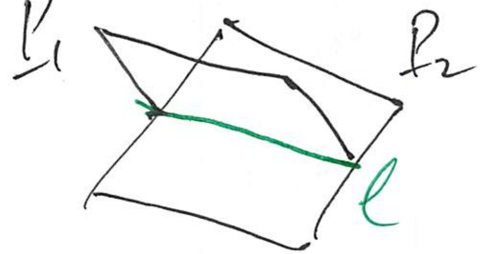
$$\Longrightarrow \quad x + y + z = D$$
$$1 - 1 + 0 = D = 0$$
$$\Longrightarrow \quad \boxed{x + y + z = 0}.$$

e.g. given 2 distinct planes,

(1) they can be parallel



(2) they can intersect in a line



e.g. $x + 2y + 3z = 7 \quad P_1$
$\quad\;\; 2x + y - z = 8 \quad P_2$

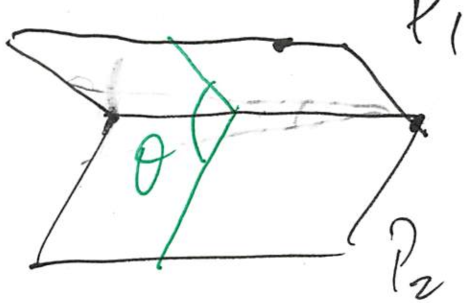
$\cap$ or $\parallel$?

planes parallel $\Longleftrightarrow$ normal vectors are nonzero scalar multiples.

$P_1 : \langle 1, 2, 3 \rangle$
$P_2 : \langle 2, 1, -1 \rangle$ $\Big\}$ not multiples

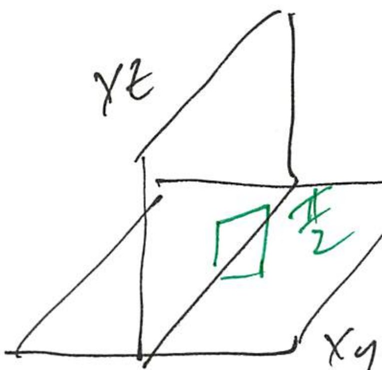
$\Longrightarrow$ intersect.

e.g. angle between 2 planes
a.k.a. "dihedral angle"



θ is maximum angle between two lines in the planes

e.g.



Same as angle between normal vectors to the planes from previous page

$\vec{n}_1 = \langle 1,2,3 \rangle$

$\vec{n}_2 = \langle 2,1,-1 \rangle$

use dot. $\vec{n}_1 \cdot \vec{n}_2 =$

$2+2-3 = 1$

$= |\vec{n}_1||\vec{n}_2| \cos\theta$

$|\vec{n}_1| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$|\vec{n}_2| = \sqrt{2^2+1^2+(-1)^2} = \sqrt{6}$

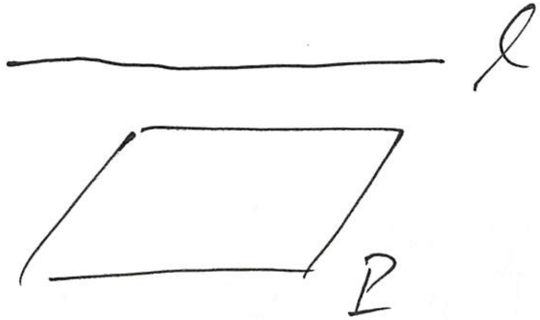
$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{84}}\right)$

<u>Lines and planes</u> l, P

① $l \cap P$ in a single point

② l contained in P

③ $\ell \parallel P$



P

⊙ If ℓ contained, or $\parallel$,
we must have dir vect $\perp$ normal vector.
If so, and have 1 pt in common, then contained.
otherwise parallel.

⊙ if dir vect $\not\perp$ normal vec,
then have pt intersection.

e.g. $\ell : \begin{cases} x = t \\ y = t \\ z = t \end{cases}$ $\vec{v} = \langle 1,1,1 \rangle$

$P : x + 2y + 3z = 1$ $\vec{n} = \langle 1,2,3 \rangle$

$\vec{n} \cdot \vec{v} \neq 0 \Rightarrow$ must $\cap$ in a pt.

Let's find this pt!
substitute eqns for ℓ in P.
$\Rightarrow$ get an eqn with t.

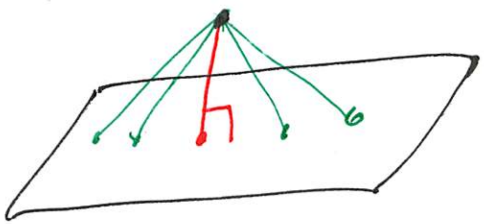
$$t + 2t + 3t = 1$$
$$6t = 1$$
$$t = \frac{1}{6}$$

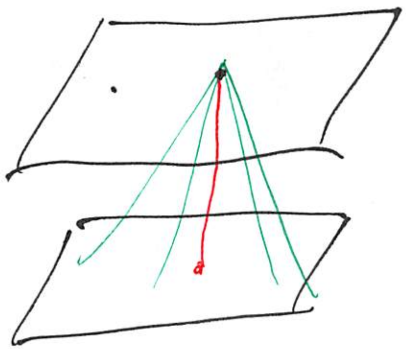
$\Rightarrow$ intersection pt is $\boxed{\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)}$

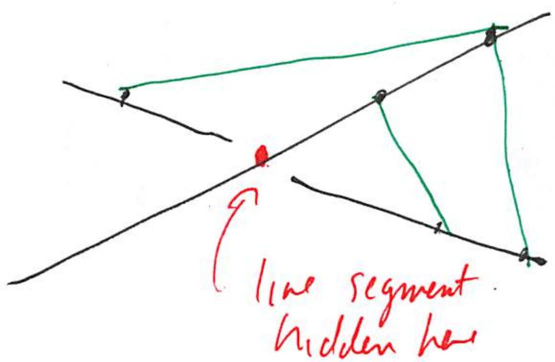
Distance problems

e.g. find distance between:
- pt and a plane
- 2 parallel planes
- a line parallel to a plane
- 2 skew lines.

"Distance" means minimum distance attained between 2 pts on the objects.







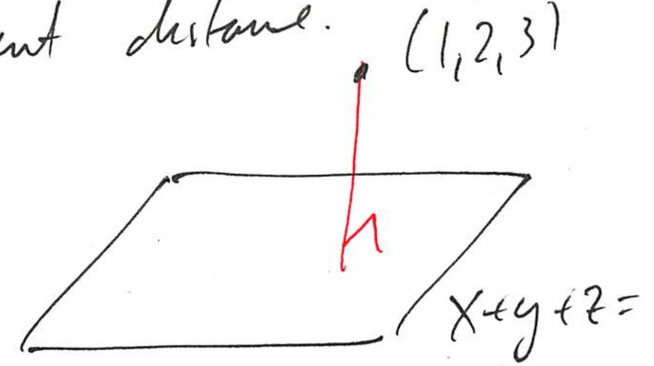
Basic technique:

1. Need a unit vector $\hat{N}$ running in the direction where distance is being measured.
2. we need an arbitrary vector $\vec{z}$ running between the 2 objects.
3. distance is $\left| \hat{N} \cdot \vec{z} \right|$

e.g. $(1,2,3)$

plane is $x+y+z=1$

want distance.



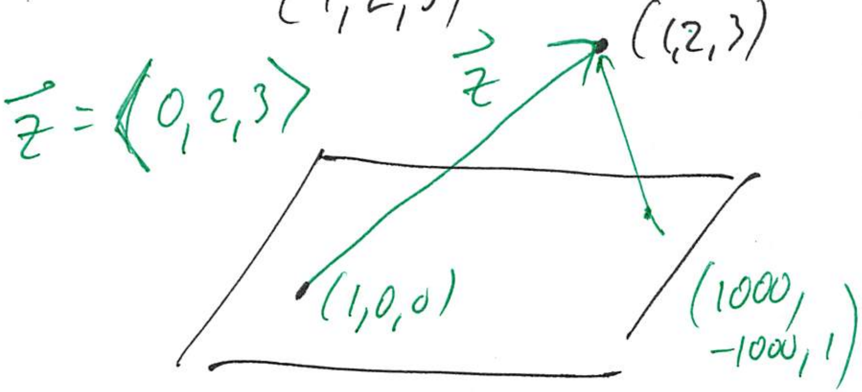
10 $\hat{N}$ is a unit vector $\perp$ to the plane.

$$\vec{n} = \langle 1,1,1 \rangle$$

$$\hat{N} = \langle 1,1,1 \rangle / \sqrt{3}$$

$$= \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$\vec{z}$: take <u>any</u> pt in plane, build a vector from it to $(1,2,3)$



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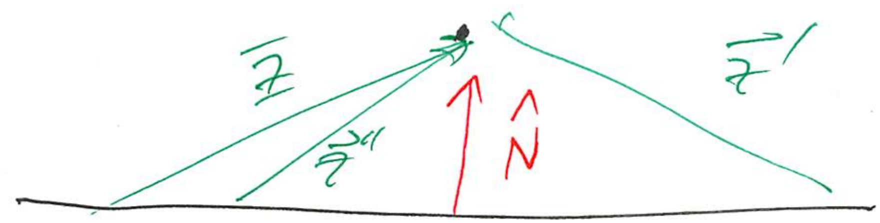
distance:

$$|\hat{N} \cdot \vec{z}|$$

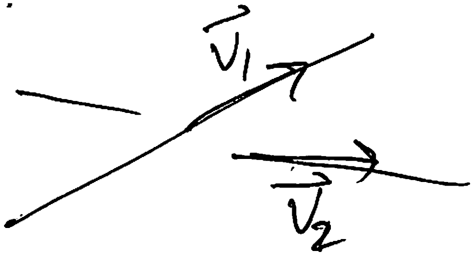
$$= \left| \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \cdot \langle 0,2,3 \rangle \right|$$

$$= \left| \frac{5}{\sqrt{3}} \right| = \frac{5}{\sqrt{3}}$$

reason: we're computing the component of $\vec{z}$ along $\hat{N}$.



e.g. 2 skew lines



use $\vec{v}_1 \times \vec{v}_2$ to make vector $\perp$ to lines.

$\vec{v} \times \vec{w} = \vec{N}$

~~make~~ it $\hat{N}$, ...

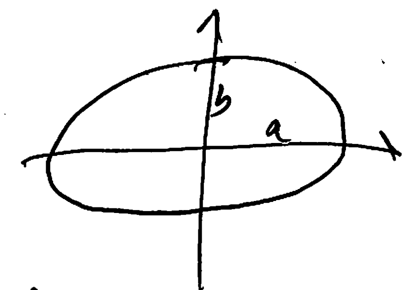
<u>Quadric ~~su~~ surfaces</u>.

special graphs in 3D come up a lot.

so far: lines
planes
spheres.

quadric surfaces are 3D analogues of "conic sections"

conic sections are ellipses, hyperbolas, parabolas, circles, ..

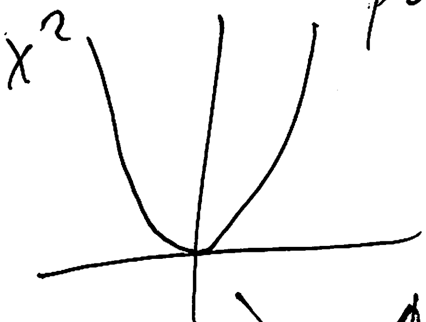


Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

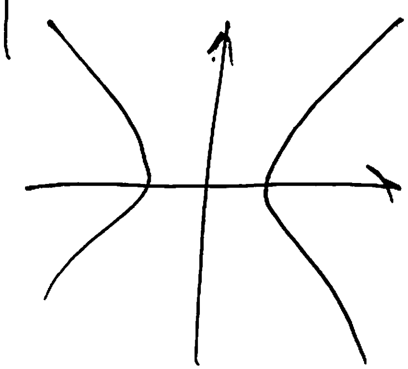
Parabola

$y = x^2$



hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Observe: highest power of a variable is 2.

e.g. $x^2 - xy + y^2 = 1$
rotated ellipse.

Want to consider similar equations in 3D.

e.g. $z = x^2 + y^2$?

to help understand the graph, use method of <u>traces</u>

aka. method of <u>slices</u>

idea: set one coordinate equal to a constant C.
Look at resulting 2D graph.
The result is a "slice" through the object parallel to one of the coordinate planes

$z = x^2 + y^2$

try $z = C$ for various C.
give intersection of our shape with plane parallel to the xy plane.

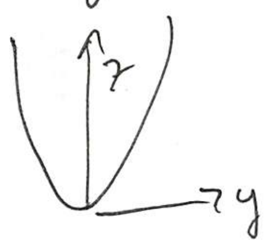
C	graph	
0	$x^2+y^2=0$	$x=y=0$ single pt
1	$x^2+y^2=1$	circle.
2	$x^2+y^2=2$	
-1	$x^2+y^2=-1$	no soln

can slice parallel to other coordinate planes

$z = x^2 + y^2$

try $x = C$.

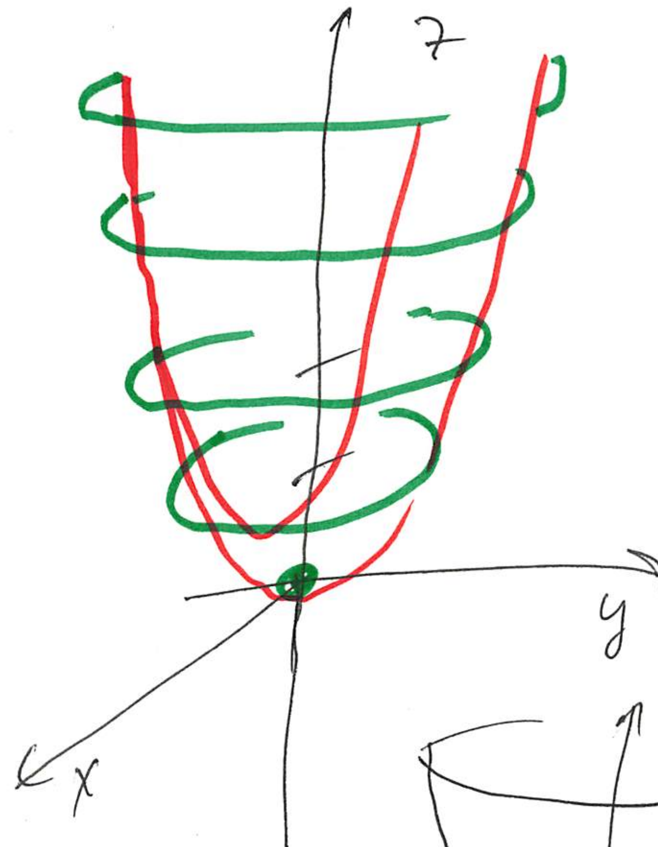
$C = 0 \qquad z = y^2$



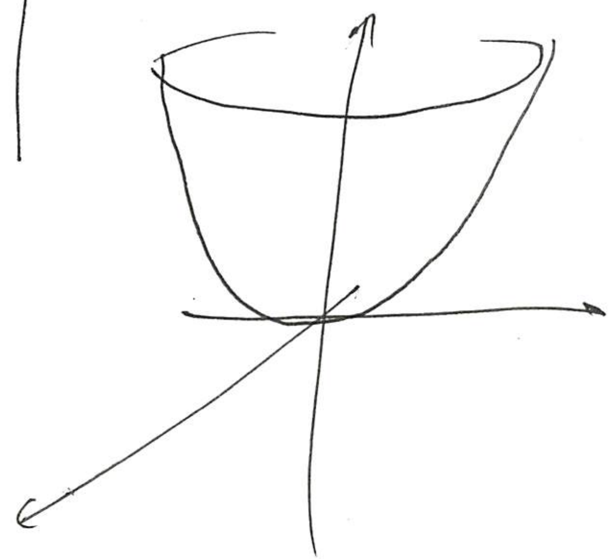
keep going

$C = 1 \qquad z = y^2 + 1$

$\boxed{\text{paraboloid}}$



parabola revolved around z axis



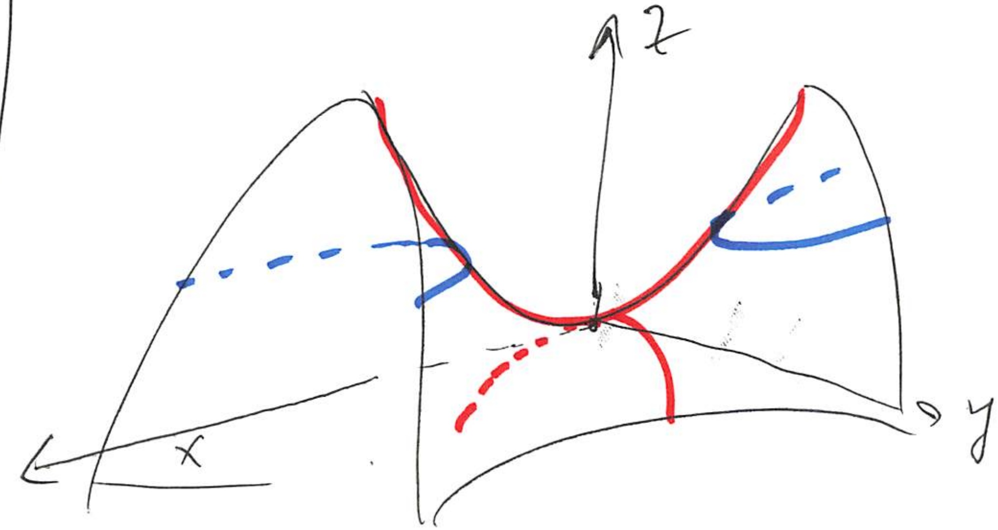
Look at more examples in book.

e.g. hyperbolic paraboloid.

This is a quadric surface that has no direct analogue in 2D

(11)

$$z = x^2 - y^2$$



"saddle shape"

hyperbola slices
parabola slices