Last time: VVFs.

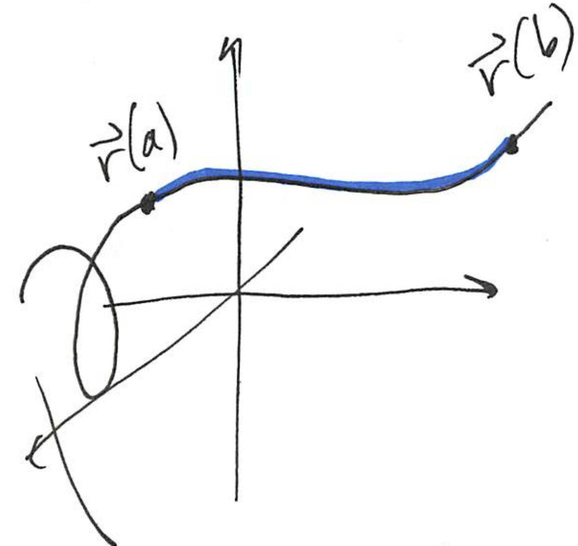
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$\int \vec{r}(t)\,dt$ = integrate components individually.

<u>Today</u> arc length
velocity / acceleration.

Arc length: goal is to measure the length of a graph of a VVF.

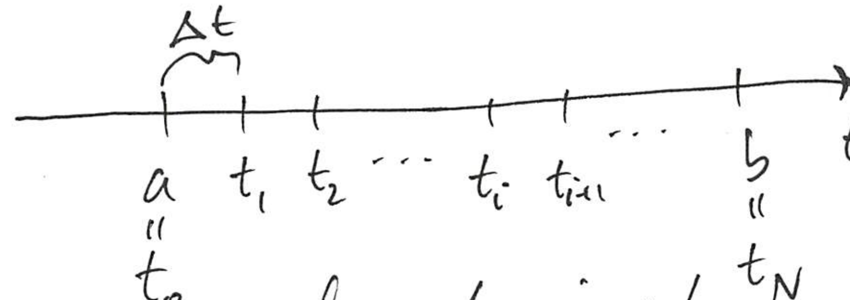


①

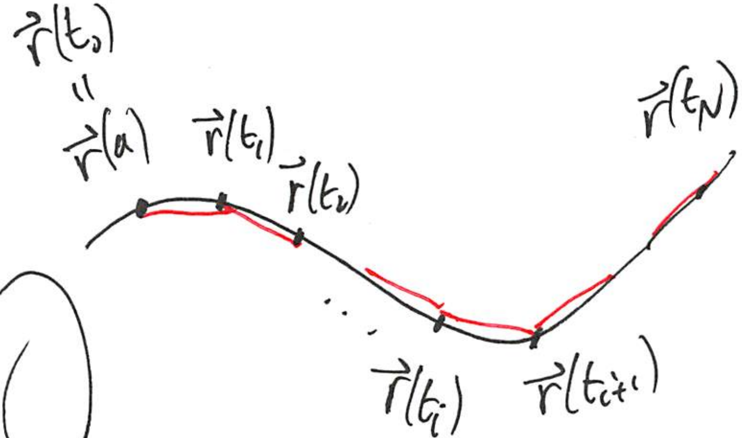
$\vec{r}(t)$
$t=a$
$t=b$

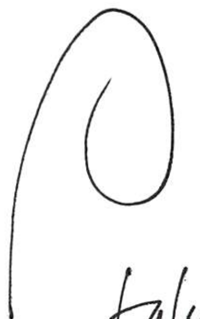
Q: how long is this section of the graph?

Idea: approximate using things we can compute the length of. use line segments



chop domain into small intervals.





take line segments

$$\vec{r}(t_{i+1}) - \vec{r}(t_i)$$

Approximation is the sum of these lengths

$$\lim_{\Delta t \to 0} \sum_{i=0}^{N-1} |\vec{r}(t_{i+1}) - \vec{r}(t_i)|$$

Claim: we get an integral

$$\int_a^b |\vec{r}\,'(t)|\,dt$$

(2)

why is this the answer?

reason: each little segment has length

$$\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

where Δx = change in x coord, etc.

$$\text{length} \approx \sum_{\text{pieces}} \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

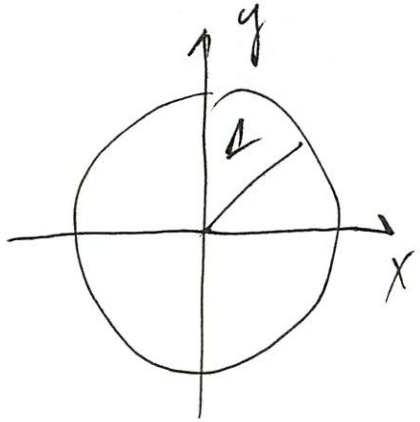
$$= \sum_{\text{pieces}} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}\,\Delta t$$

take limit $\Delta t \to 0$

$$\int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$\Rightarrow \int_a^b |\vec{r}\,'(t)|\,dt$$

e.g. circumference of circle



$\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$

This traces out the circle once.

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t}$

$= 1$

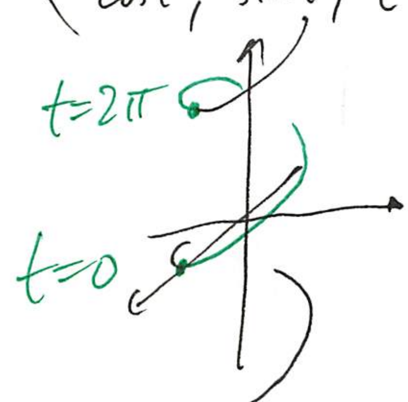
$\Longrightarrow$ length is

$$\int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$$

correct because circle of radius a has circumf. $2\pi a$.

e.g. helix length of helix between $t=0$, $t=2\pi$

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$



$$\text{ans} = \int_0^{2\pi} |\vec{r}\,'(t)|\,dt$$

$$= \int_0^{2\pi} |\langle -\sin t, \cos t, 1\rangle|\,dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\,dt$$

$$= \int_0^{2\pi} \sqrt{2}\,dt$$

$$= \sqrt{2}\,t\Big|_0^{2\pi} = \boxed{2\sqrt{2}\pi}$$

e.g. $\vec{r}(t) = \langle e^t, e^{-t}, t\sqrt{2}\rangle \quad 0 \le t \le 1$

$$\int_0^1 |\vec{r}\,'(t)|\,dt$$

$$\vec{r}\,'(t) = \langle e^t, -e^{-t}, \sqrt{2}\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{e^{2t} + e^{-2t} + 2}$$

$$\int_0^1 \sqrt{e^{2t} + e^{-2t} + 2}\,dt.$$

Ideas:
- complete □
- u-subst $u = e^t, \ldots$
- mult / divide by e^{2t}
- convert to trig

Claim: it's a perfect square under the $\sqrt{}$

$$e^{2t}+e^{-2t}+2 = (e^t+e^{-t})^2$$

$$\Rightarrow \int_0^1 (e^t+e^{-t})\,dt$$

(ok because $e^x > 0$)

$\boxed{\sqrt{a^2} \neq a \quad = |a|}$

$$= e^t - e^{-t}\Big|_0^1$$

$$= e - e^{-1} - (1-1)$$

$$= \boxed{e - \frac{1}{e}}$$

e.g. $\vec{r}(t) = \langle 12t, 8t^{3/2}, 3t^2 \rangle$ 1

$0 \le t \le 1$.

$$\vec{r}'(t) = \langle 12, 12t^{1/2}, 6t \rangle$$

$$|\vec{r}'(t)| = \sqrt{144 + 144t + 36t^2}$$

$$= \sqrt{36(4+4t+t^2)}$$

$$= 6\sqrt{(t+2)^2}$$

since $t+2 \ge 0$ on $0 \le t \le 1$, we get $6(t+2)$

Length $\int_0^1 6(t+2)\,dt = 6\left(\frac{1}{2}t^2+2t\right)\Big|_0^1$

$$= \boxed{15}$$

Velocity / acceleration

VVF $\vec{r}(t)$

think of it as modeling the position of a particle in space.

$\vec{r}(t)$ position as a function of time t

$\vec{r}'(t) = \frac{d\vec{r}}{dt}$ rate of change of position as a function of time

This is the velocity of the particle

$\vec{v}(t)$

This is itself a VVF. it has both direction and magnitude.

$|\vec{r}'(t)| = |\vec{v}(t)|$ is called the <u>speed</u>.

speed ≥ 0.

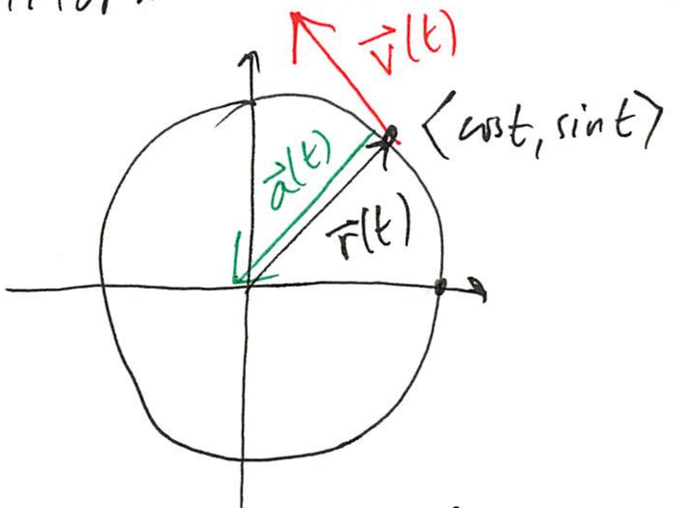
$\vec{r}''(t) = \frac{d\vec{v}}{dt} =$ rate of change of velocity with respect to time.

$=$ acceleration

$= \vec{a}(t)$

e.g. $\vec{r}(t) = \langle \cos t, \sin t \rangle$.

"uniform circular motion"



$\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$

$\Rightarrow$ speed is constant.
(speed is uniform)

$\vec{v}(t)$ isn't constant.

even though speed is constant, direction is changing.

also $\vec{v}(t)$ is tangent to the graph at $\vec{r}(t)$.

claim: $\vec{v}(t) \perp \vec{r}(t)$ for all t (for this example)

check:

$$\vec{v} \cdot \vec{r} = \langle -\sin t, \cos t \rangle \cdot \langle \cos t, \sin t \rangle = 0 \checkmark$$

$\vec{a}(t) = \vec{v}\,'(t) = \langle -\cos t, -\sin t \rangle$

observe: $\vec{a} \perp \vec{v}$.

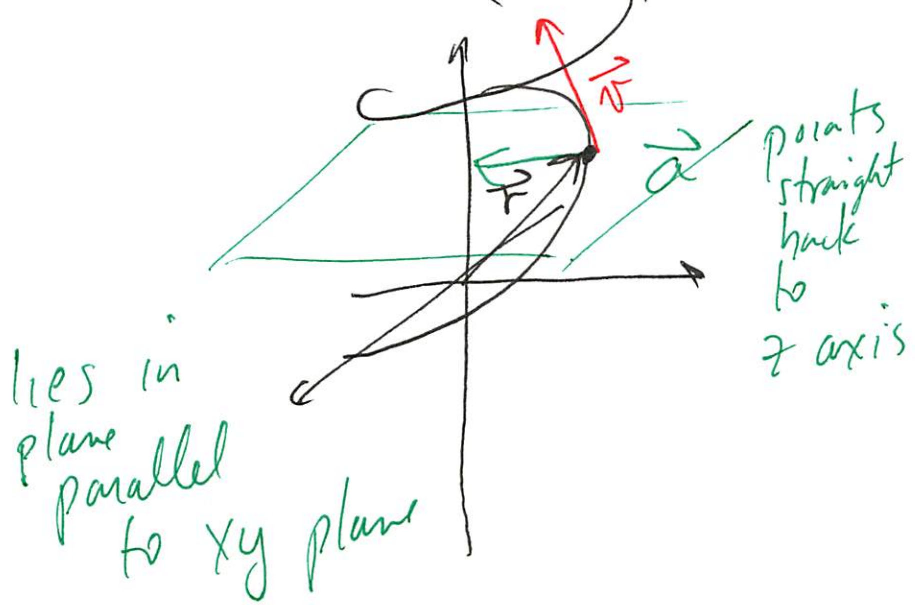
e.g. helix. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$= \langle \cos t, \sin t, 0 \rangle + \langle 0, 0, t \rangle$

$\vec{r}\,'(t) = \vec{v}(t) = \langle -\sin t, \cos t, 1 \rangle$

$|\vec{v}(t)| = \sqrt{2}$ cf. arc length.

$\vec{a}(t) = \vec{v}\,'(t) = \langle -\cos t, -\sin t, 0 \rangle$



e.g. have moving particle and we know

$\vec{r}(0) = \langle 1, 0, 0 \rangle$ initial pos.

$\vec{v}(0) = \langle 1, -1, 1 \rangle$ initial vel.

$\vec{a}(t) = \langle 4t, 6t, 1 \rangle$.

find $\vec{v}(t)$, $\vec{r}(t)$.

ans: integrate twice, incorporating initial conditions.

$$\vec{v}(t) = \int \vec{a}(t)\,dt$$

$$= \int \langle 4t, 6t, 1 \rangle \,dt$$

$$= \langle 2t^2, 3t^2, t \rangle + \vec{C}$$

use $\vec{v}(0) = \langle 1, -1, 1 \rangle$.

$$\Longrightarrow \vec{C} = \langle 1, -1, 1 \rangle.$$

$$\rightarrow \vec{v}(t) = \langle 2t^2+1, 3t^2-1, t+1 \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

$$= \int \langle 2t^2+1, 3t^2-1, t+1 \rangle\,dt$$

$$= \left\langle \frac{2}{3}t^3+t,\ t^3-t,\ \frac{t^2}{2}+t \right\rangle + \vec{C}$$

use $\vec{r}(0) = \langle 1, 0, 0 \rangle$.

$$\Rightarrow \vec{C} = \langle 1, 0, 0 \rangle$$

Ans:

$$\vec{r}(t) = \left\langle \frac{2}{3}t^3+t+1,\ t^3-t,\ \frac{t^2}{2}+t \right\rangle$$