last time: gradient

$f(x,y)$ $\quad \nabla f = \langle f_x, f_y \rangle$

$g(x,y,z)$ $\quad \nabla g = \langle g_x, g_y, g_z \rangle$

$D_{\hat{u}}(f)$ = directional derivative of f in direction of $\hat{u}$

$= (\nabla f) \cdot \hat{u}$

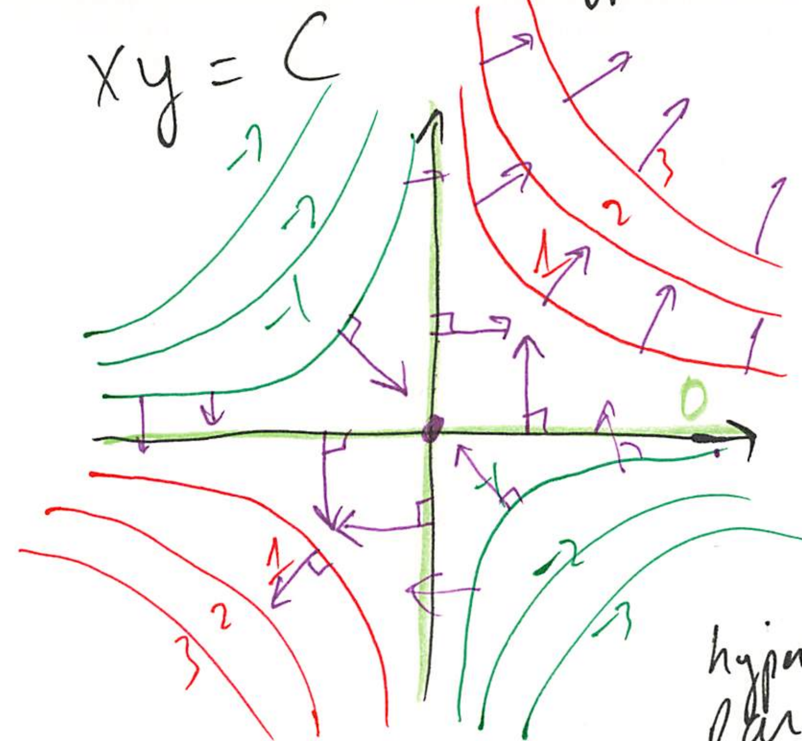
∇f is $\perp$ to level curves of f

points in direction of <u>steepest</u> <u>ascent</u>

e.g. $f(x,y) = xy$

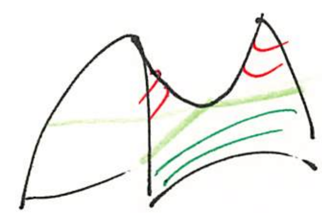
level curves are hyperbolas.

$xy = C$



hyperbolic parabol.

$\nabla f = \langle y, x \rangle$



②

e.g. $g(x,y,z)$ have level surfaces.

graphs of $g(x,y,z) = C$

∇g is $\perp$ to these. Can use this to get eqns of tangent planes.

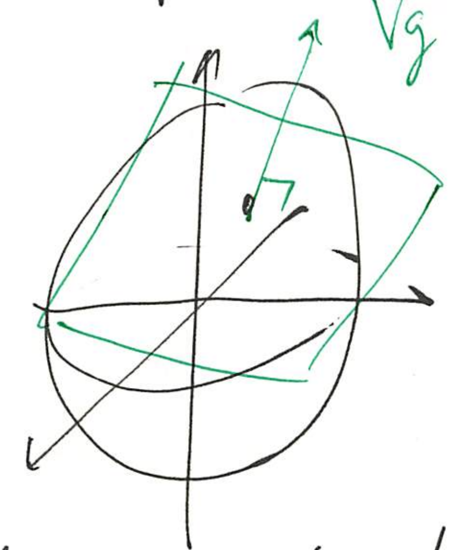
e.g. sphere. $g(x,y,z) = x^2+y^2+z^2$

$g=1$ unit sphere.

$\nabla g = \langle 2x, 2y, 2z \rangle$

if (x,y,z) is on the sphere, then

$\langle 2x, 2y, 2z \rangle \perp$ sphere at that point.



find eqn of tangent plane at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

normal vector is $\nabla g = \langle 2/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3} \rangle$

$2/\sqrt{3}\,x + 2/\sqrt{3}\,y + 2/\sqrt{3}\,z = D$

$\Rightarrow D = 2$

So tangent plane is

$$\frac{2}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y + \frac{2}{\sqrt{3}}z = 2$$

e.g. If $z = f(x,y)$

think of it as

$$g(x,y,z) = f(x,y) - z$$

then original graph is the level surface $g(x,y,z) = 0$

$\Rightarrow \nabla g \perp$ graph.

$$\nabla g = \langle f_x, f_y, -1 \rangle$$

is $\perp$ to graph

e.g. $f(x,y) = x^2 - y^2$

$(2, 1, 3)$

to get the tangent plane, (3)

the vector $\langle f_x, f_y, -1 \rangle$ is $\perp$ to graph.

$$\langle 2x, -2y, -1 \rangle \Big|_{(2,1,3)} = \langle 4, -2, -1 \rangle$$

T.P.: $4x - 2y - z = D$

plug in $(2,1,3)$ get $D = 3$

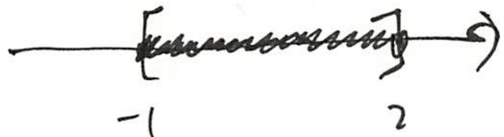
$$\boxed{4x - 2y - z = 3}$$

§14.7 Max/Min

$y = f(x)$ have 2 kinds of max/min probs

(1) Max/Min on closed/bounded domain

e.g. find max/min of $f = x^2$
on $[-1, 2]$ $\quad -1 \le x \le 2$



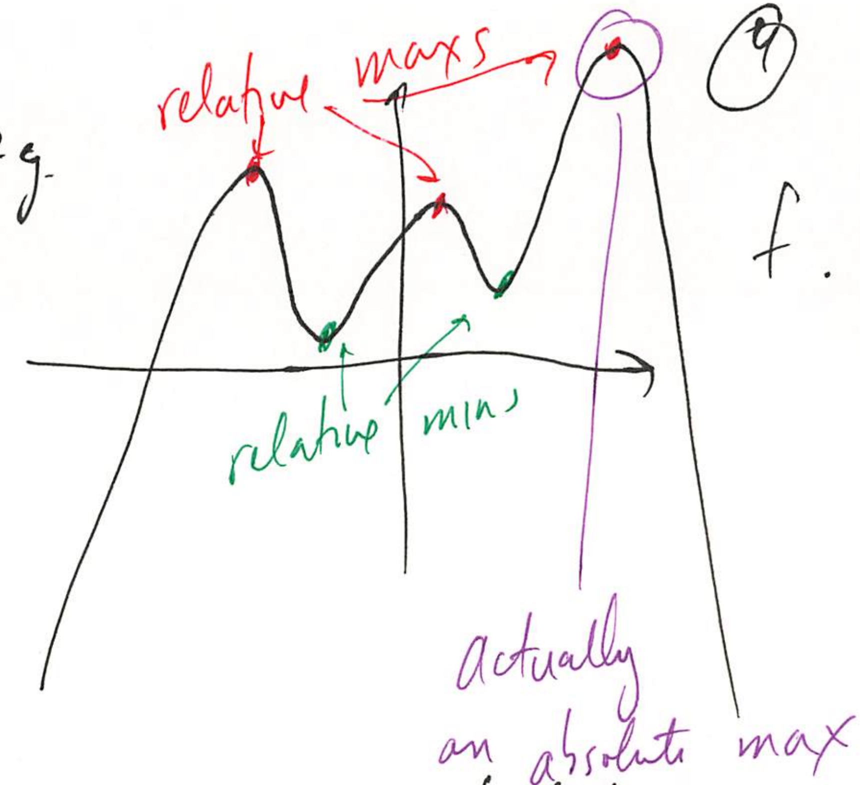
we want <u>absolute</u> max/mins

② max/min on domain that's not closed and bounded.

e.g find max/min of $f = x^2$ on real line

here we look for <u>relative</u> max/mins

e.g.



There is no absolute min for f.

general domain: there is no guarantee that abs max/min exists.

closed/bounded domain: guaranteed to have an abs max/min

To find Max/Mins in the 2 cases.

(1) closed/bounded.

1st find candidates for abs mx/mn.

(a) critical pts: $f'=0$ or f' not defined.

(b) endpts of the domain.

2nd: evaluate f on candidates to find biggest smallest value.

e.g. $[-1,2]$, $f=x^2$, $f'=2x$

C.P. $x=0$

endpts $x=-1,2$.

Cands: $x=-1,0,2$.

$f(-1)=1$

$f(0)=0$ MN (Abs)

$f(2)=4$ MX (Abs)

(2) general domain.

1st find candidates.. just critical pts

2nd: classify crit pts.

1st derivative test

2nd derivative test

Now fns of more than 1 variable

Now a critical pt $\nabla f=\vec{0}$.

$\vec{\nabla} f = \vec{0} \iff$ all partial derivatives vanishing

e.g. $f(x,y) = 4x+6y-x^2-y^2$

$\nabla f = \langle 4-2x, 6-2y \rangle \overset{?}{=} \vec{0}$

$\Longrightarrow \begin{cases} 4-2x=0 \\ 6-2y=0 \end{cases}$

$x=2, y=3$

$(2,3)$ only crit pt.

e.g. $f = x^2+xy+y^2-2x+3$

$\nabla f = \langle 2x+y-2, x+2y \rangle$

$= \vec{0}$

$\begin{cases} 2x+y=2 \\ x+2y=0 \end{cases}$

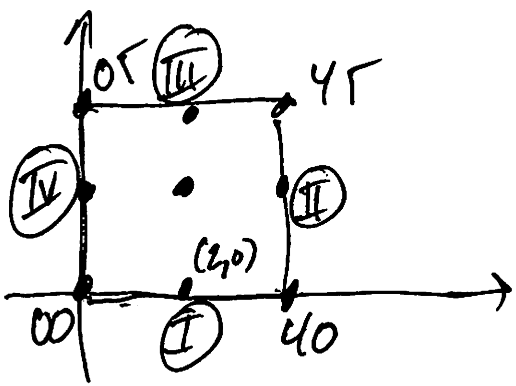
$x = 4/3, \ y = -2/3$

closed / bounded domain

1. find candidates:
   a. critical pts in interior
   b. critical pts on boundary
   c. corner pts
2. classify by plugging into f.

$f=$

e.g. Abs max/min of $4x+6y-x^2-y^2$ on closed rectangle with vertices $(0,0)$, $(4,0)$, $(4,5)$, $(0,5)$



I $y=0$

II $x=4$

III $y=5$

IV $x=0$

critical pts : ⓐ $\nabla f=\vec{0} \Leftrightarrow (2,3)$

ⓒ corner pts

ⓑ edges.

use the equations holding on the edge to reduce the problem to a 1-variable max/min. look for critical pts as in Calc 1.

I : $y=0$ $f(x)=4x-x^2$

$f'(x)=4-2x\overset{?}{=}0$

$x=2$

$$f(x,y) = 4x+6y-x^2-y^2$$

Candidates

(2,3)
(0,0)
(4,0)
(4,5)
(0,5)
(2,0)
(4,3)
(2,5)
(0,3)

II $x=4$

$f(y)=6y-y^2$

$f'(y)=6-2y$ $y=3$

III $4x-x^2+5=f$

$y=5$ $f'=4-2x$ $x=2$.

IV $x=0$

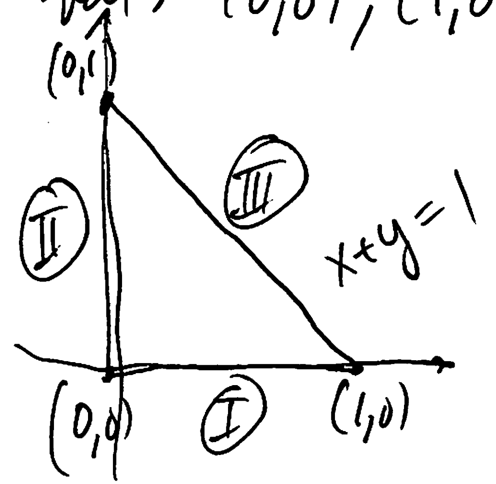
$f=6y-y^2$

$f'=6-2y$ $y=3$

take these pts, plug into $f(x,y)$ and find the largest/smallest values.

try it!

e.g. what if domain is triangle with verts $(0,0)$, $(1,0)$, $(0,1)$



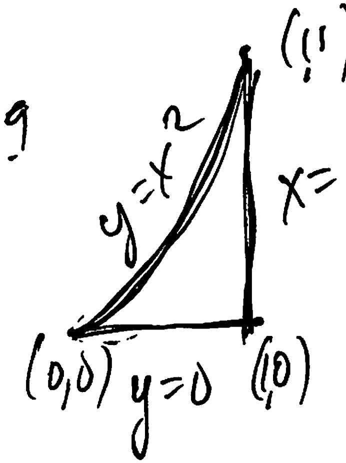
I $y=0$

II $x=0$

III $x+y=1 \Rightarrow y=1-x$

so substitute $y=1-x$ into f, just have 1 variable.

e.g.



subst $y=x^2$ to deal with parabola edge.

---

for general domain, have analogue of 2nd derivative test!

$$\underbrace{f_{xx},\ f_{xy}=f_{yx},\ f_{yy}}_{\text{all contain info we need.}}$$

We use them in Hessian

$$D = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$= f_{xx} f_{yy} - (f_{xy})^2$$

e.g. $f = 4x + 6y - x^2 - y^2$

$f_x = 4 - 2x$

$f_y = 6 - 2y$

$f_{xx} = -2$

$f_{yy} = -2$

$f_{xy} = f_{yx} = 0$

$$D = \det \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = 4$$

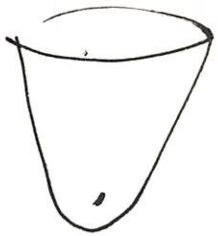
2nd Derivative test

compute D at a critical pt

1. if $D > 0$,
   a. $f_{xx} > 0 \Rightarrow$ Rel Min
   b. $f_{xx} < 0 \Rightarrow$ Rel Max
2. if $D < 0$, saddle point
3. if $D = 0$, test fails.

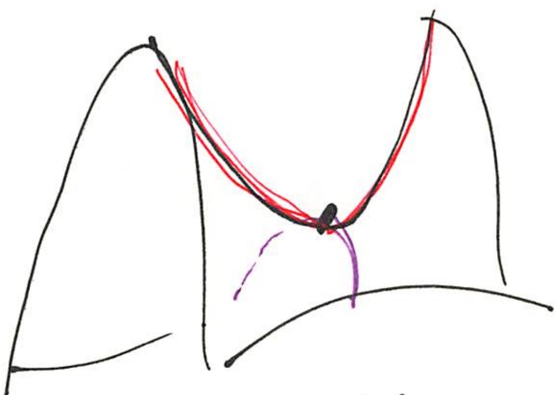
1 (a)

(b)



looks like paraboloids.

2



looks like origin of hyp. parab.

simultaneously appears to be a max and a min

Can't happen in Calc 1.