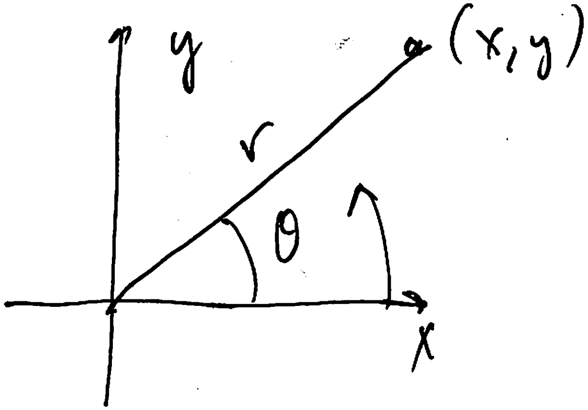
Multiple integration : polar coordinates



r, θ

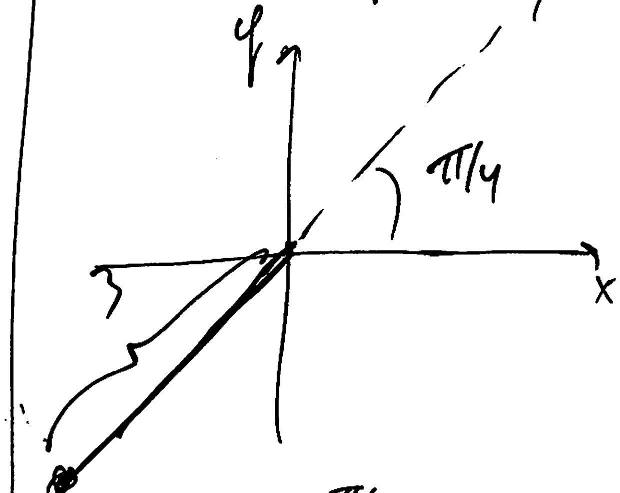
$$r = \sqrt{x^2+y^2}$$

$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & x \neq 0 \\ \pi/2,\ 3\pi/2 & x = 0,\ y \neq 0 \end{cases}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

We allow $r < 0$.
Find angle first, then go in negative direction to measure r.

$r = -3,\ \theta = \pi/4$



$r = -3,\ \theta = \pi/4$

r	θ
-3	$\frac{\pi}{4}$
3	$\frac{5\pi}{4}$
-3	$\frac{9\pi}{4}$

Beware: in polar coords, the same point in the plane can have multiple names.

r, θ same as $r, \theta + 2\pi$

double integrals in polar

$$\iint_R f \, dA \qquad 3 \text{ components}$$

① f function

② R region = domain of integration.

③ dA tiny element of area in R.

① if have $f(x,y)$, we can subst $x = r\cos\theta$, $y = r\sin\theta$.

$f(r\cos\theta, r\sin\theta)$ is a function in r, θ

② region: we need to describe regions in plane using polar coordinates. Start with simplest regions: analogues of rectangles in x, y.

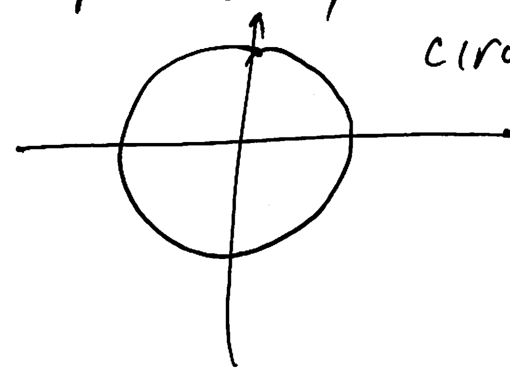
$$\left.\begin{array}{l} a \le x \le b \\ c \le y \le d \end{array}\right\} \text{limits are constants independent of } x, y.$$

We should consider

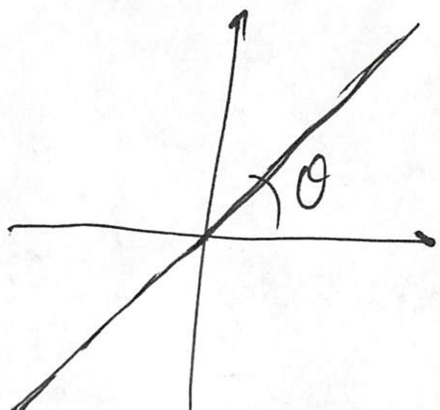
$$a \le r \le b$$
$$c \le \theta \le d$$

e.g. $r = \text{const}$, θ free

circle of radius r.



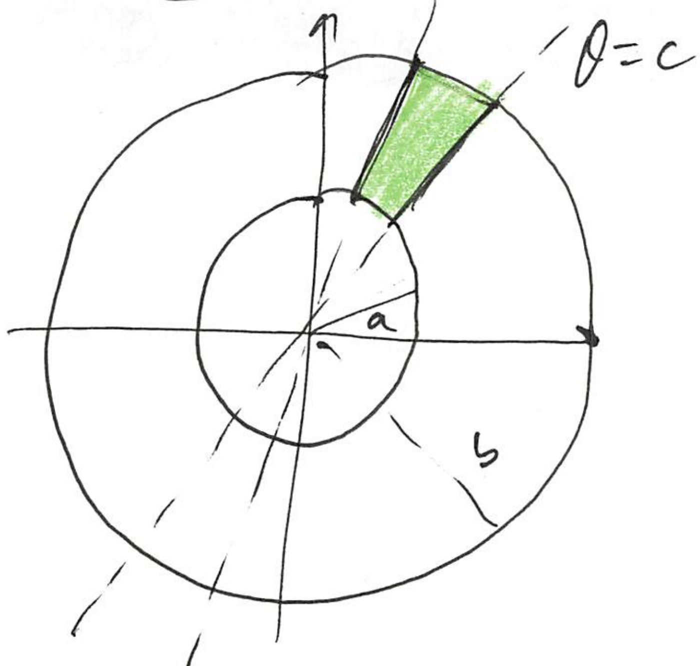
e.g. θ fixed, r free.



get line through the origin.

e.g.

$$a \le r \le b$$
$$c \le \theta \le d$$



(3) $dA = ?$

in xy,

$$dA = dx\,dy$$

in polar,

$$\cancel{dA = dr\,d\theta}$$

incorrect.

dr units of length

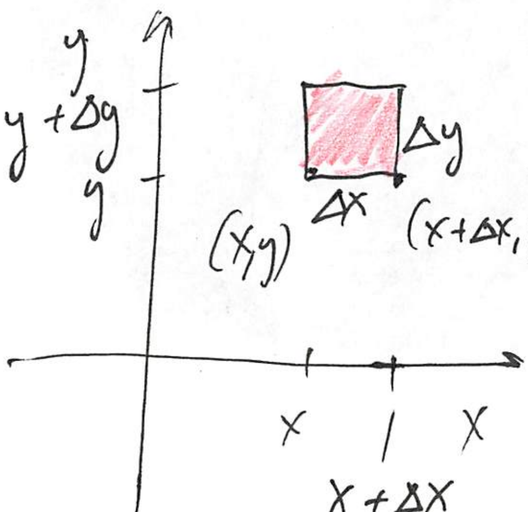
$d\theta$ dimensionless quantity doesn't have units of length

$\Rightarrow$ dA would have units of length, not length^2.

claim:

$dA = r\,dr\,d\theta$ ✓

observe: have an extra r, and it can't be omitted. it must be in the integral.

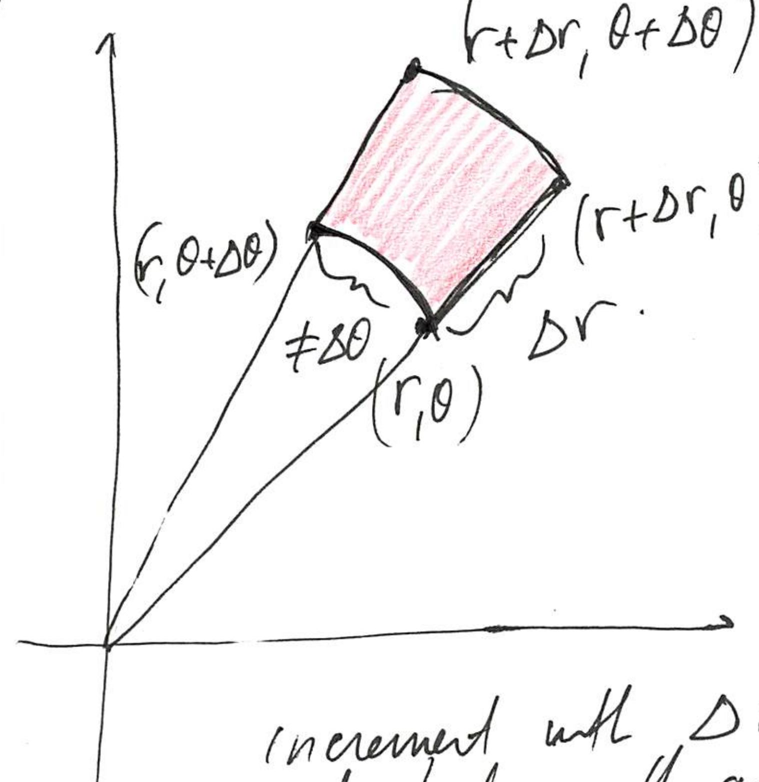


$\Delta A = \Delta x \Delta y$

take limit and

$\Delta A \to dA$

get $dx\,dy$



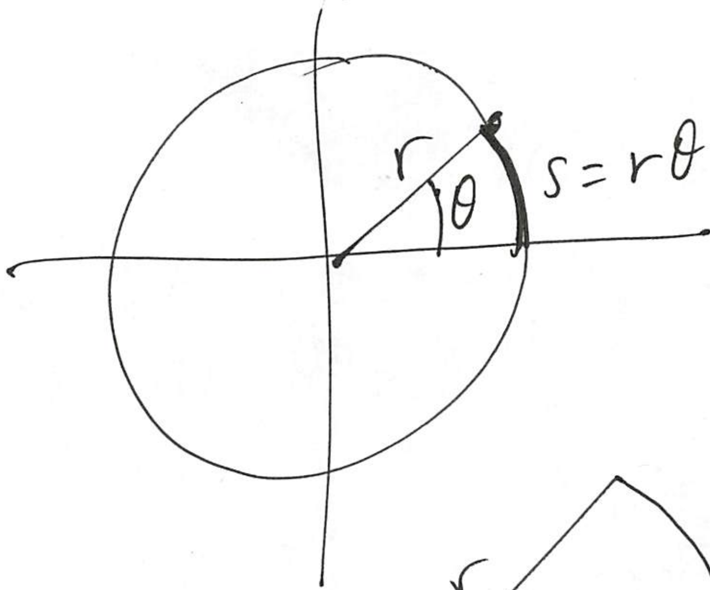
increment with $\Delta r, \Delta\theta$ get shape with area

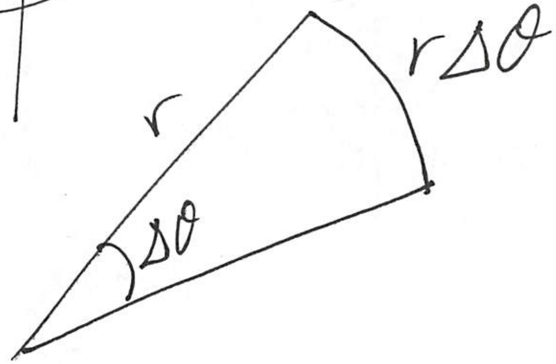
ΔA

2 sides are curved. top and bottom are basically same length. (relative to size of $\Delta r, \Delta\theta$)

bottom distance is

$r\Delta\theta$





$$\Longrightarrow \quad \Delta A = r\,\Delta r\,\Delta\theta$$

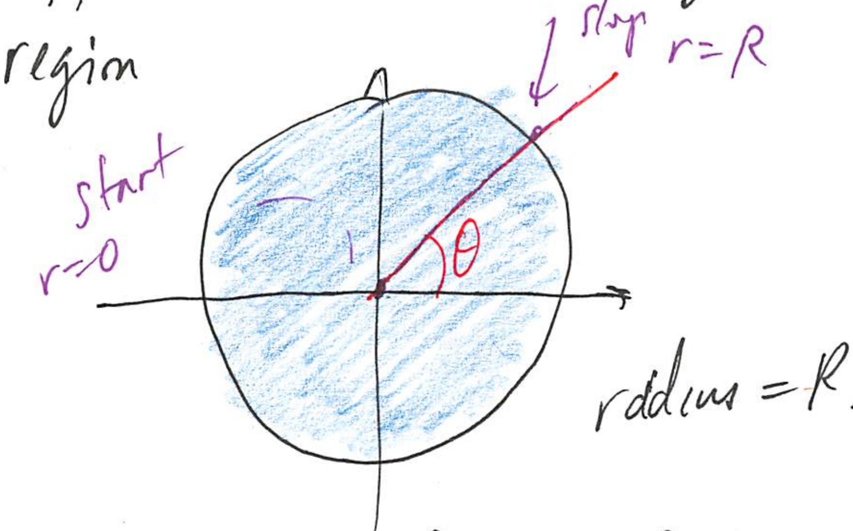
take limit: $\Delta A \to dA$

$$= r\,dr\,d\theta\,.$$

(1)

e.g. area of circle of radius R.

$$\iint_{\text{region}} dA = \text{area of region}$$



$$0 \le \theta \le 2\pi$$

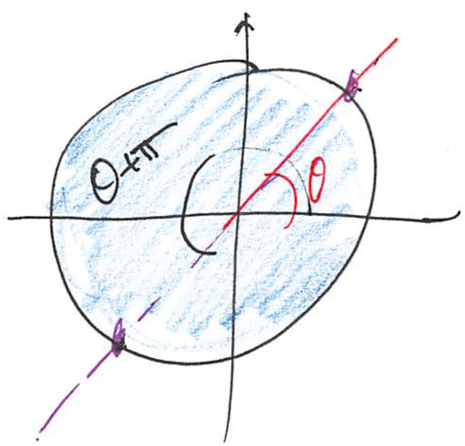
$$0 \le r \le R$$

$$\iint_{\text{disk}} dA = \int_0^{2\pi}\int_0^R r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^R \, d\theta$$

$$= \int_0^{2\pi} \frac{R^2}{2}\, d\theta = \frac{R^2}{2}\int_0^{2\pi} d\theta$$

$$= \frac{R^2}{2}\, \theta \Big|_0^{2\pi} = \boxed{\pi R^2}$$



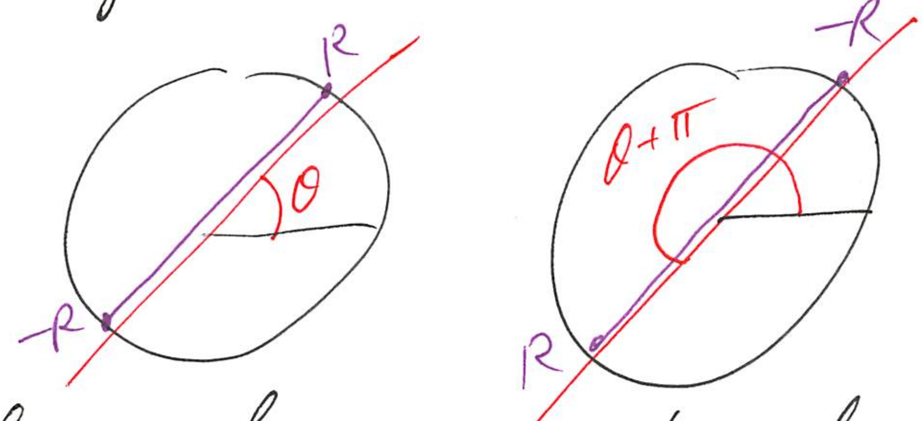
why not

$-R \le r \le R$

could do this

but then must take

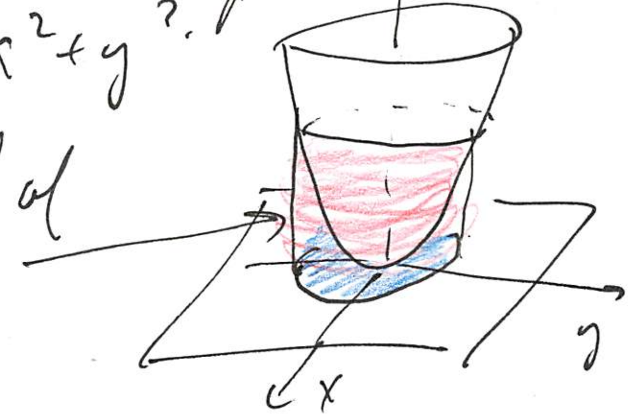
$0 \le \theta \le \pi$ otherwise we get $2\times$ what we want



e.g. volume over unit circle and under the paraboloid

$z = x^2 + y^2$.

want vol of this



Notice: Base region works well in polar coords. $0 \le \theta \le 2\pi$, $0 \le r \le 1$

We want

$$\iint_{\text{Region}} f \, dA = \iint_{\text{Region}} (x^2+y^2) \, dA$$

and the function x^2+y^2 is also nice in polar:

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

so get

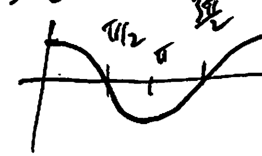
$$\int_0^{2\pi}\int_0^1 r^2 (r\,dr\,d\theta)$$

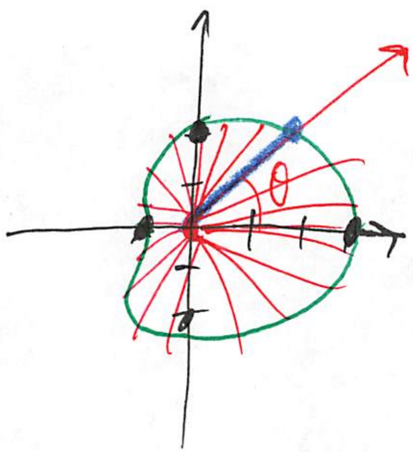
$$= \int_0^{2\pi} \frac{1}{4}r^4 \Big|_0^1 \, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} d\theta = \boxed{\frac{\pi}{2}}$$

e.g. area of cardioid $r = 2 + \cos\theta$

θ	r
0	3
$\frac{\pi}{2}$	2
π	1
$\frac{3\pi}{2}$	2





$r = 2 + \cos\theta$

$$\iint_{\text{region}} dA = \iint_{\text{region}} r\,dr\,d\theta$$

$$0 \le \theta \le 2\pi$$

$$0 \le r \le 2 + \cos\theta$$

$$\int_0^{2\pi} \int_0^{2+\cos\theta} r\,dr\,d\theta$$

8

$$= \int_0^{2\pi} \frac{1}{2} r^2 \Big|_0^{2+\cos\theta} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (2+\cos\theta)^2\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (4 + 4\cos\theta + \cos^2\theta)\, d\theta$$

use $\frac{1}{2}$ angle formula to deal with this.

. . .

e.g. volume of a cone base radius R, height h.

take disk of radius R to be the base.

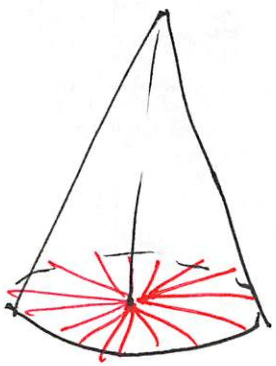
polar. $dA = r\,dr\,d\theta$

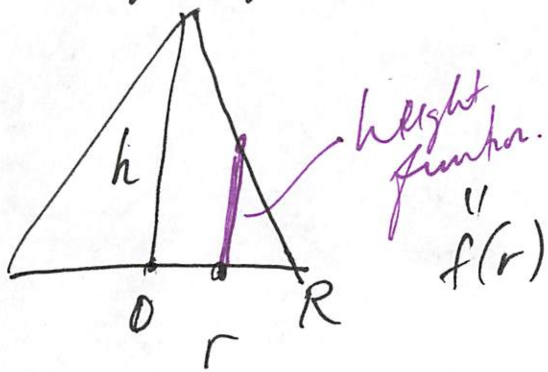
$0 \le \theta \le 2\pi$

$0 \le r \le R$

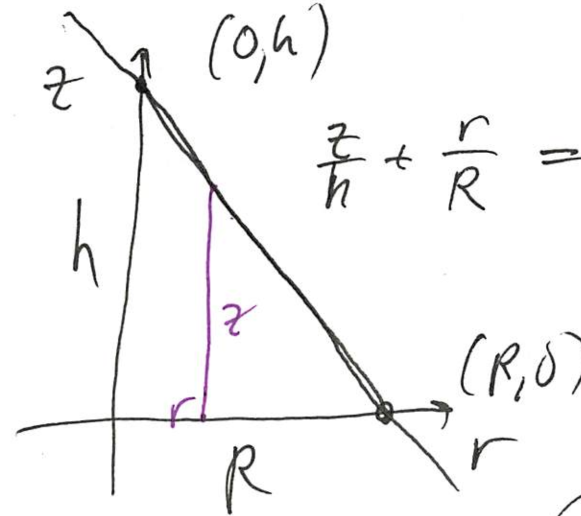
need height function

Side view





$f(r)$ only depends on r, not on θ.



$$\frac{z}{h} + \frac{r}{R} = 1$$

$$\Longrightarrow \quad z = \underbrace{h\left(1 - \frac{r}{R}\right)}_{f(r)}$$

$\Longrightarrow$ volume =

$$\iint_{\text{base}} h\left(1 - \frac{r}{R}\right) r\,dr\,d\theta$$

$$= h\int_0^{2\pi}\int_0^R \left(1 - \frac{r}{R}\right) r\,dr\,d\theta$$

Ans: $\frac{\pi h R^{\cancel{3}\,2}}{3}$

$= \frac{1}{3} \cdot h \cdot \pi R^2$

General fact:
~~a~~ volume of
~~big~~ cone over a region
is
$\frac{1}{3}$(height)(area region)

next time:
— applications of double integrals
— surface integrals