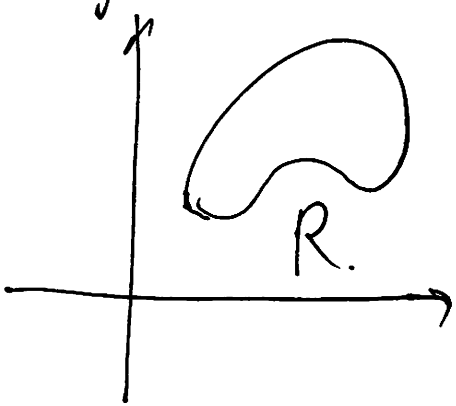
①

Some applications of double integrals

mass
centers of mass / centroids

① Mass :

R region in the plane gives a "lamina," i.e. a very thin solid.



It has thickness, but very small compared to rest

assume have density function $\rho : R \to \mathbb{R}$.

i.e. $\rho(x,y)$ that gives the density of the material at position x,y.

claim $\iint_R \rho(x,y)\,dA$ gives the mass of the region.

dA tiny piece of area.
$\rho(x,y)$ have density at (x,y)

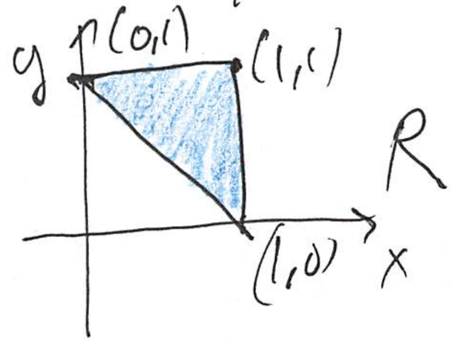
units of ρ are $\frac{\text{mass}}{\text{area}}$.

so $\rho \, dA \approx$ mass of tiny portion of the region at (x,y)

$\iint_R \rho \, dA$ = sum of these contributions to get total mass.

if density is constant K, then get $K \,\text{area}(R)$

e.g.



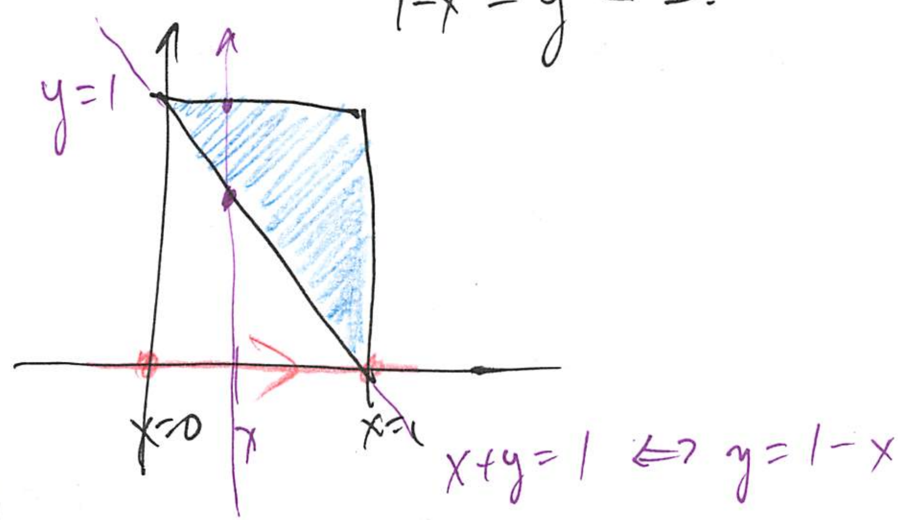
say $\rho = xy$.

$$\iint_R xy \, dA.$$

$dA = dy\,dx$

$0 \le x \le 1$

$1-x \le y \le 1.$



$$\int_0^1\int_{1-x}^1 xy\,dy\,dx = \int_0^1 x\left(\int_{1-x}^1 y\,dy\right)dx$$

$$= \frac{1}{2}\int_0^1 x\left(y^2\Big|_{1-x}^1\right)dx$$

$$= \frac{1}{2}\int_0^1 \cancel{x}\left(1-(1-x)^2\right)dx$$

$$= \frac{1}{2}\int_0^1 x\left(1-(1-2x+x^2)\right)dx$$

$$= \frac{1}{2}\int_0^1 2x^2 - x^3\,dx$$

$$\cancel{= \frac{1}{2}\left(x^2-\frac{1}{3}x^3\right)\Big|_0^1}$$

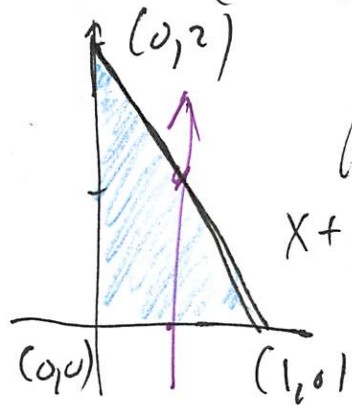
$$\cancel{= \frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{2}{3}x^3-\frac{1}{4}x^4\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{2}{3}-\frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{8-3}{12}\right) = \boxed{\frac{5}{24}}$$

2



$\rho(x,y) = 1+3x+y.$

$$x+\frac{y}{2}=1 \Rightarrow 2x+y=2$$

$$y = 2-2x$$

$$\iint_R \rho\,dA \qquad \begin{aligned} 0 &\le x \le 1 \\ 0 &\le y \le 2-2x \end{aligned}$$

$$\int_0^1 \int_0^{2-2x} (1+3x+y)\,dy\,dx$$

$$= \int_0^1 \left(y + 3xy + \frac{1}{2}y^2\right)\Big|_0^{2-2x} dx$$

$$= \int_0^1 2-2x + 3x(2-2x) + \frac{1}{2}(2-2x)^2\,dx$$

$$= \int_0^1 2-2x+6x-6x^2+2(1-2x+x^2)\,dx$$

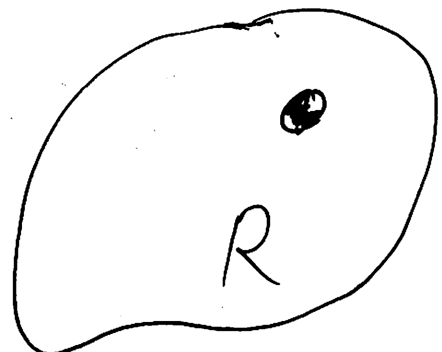
$$= \int_0^1 (-4x^2+4)\,dx$$

$$= 4\int_0^1 1-x^2\,dx$$

$$= 4\left(x - \frac{1}{3}x^3\right)\Big|_0^1$$

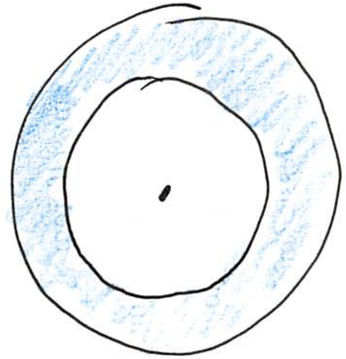
$$= \boxed{\frac{8}{3}}$$

center of mass.



ρ density

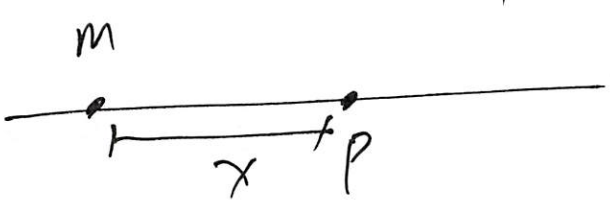
unique point at which the region R exactly balances if supported there

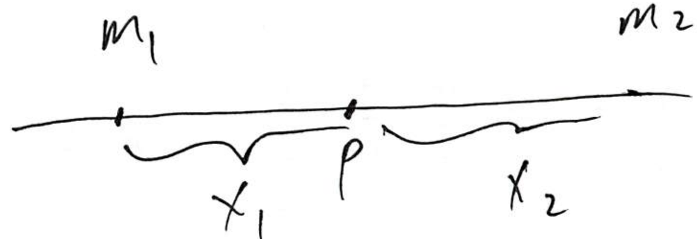
Center of mass need not be a point in the region e.g.  $\rho = \text{const}$

Special case when density is constant, ctr of mass is called the <u>centroid</u>.

Recall
Moment of a mass m about a point p is mx where x is the distance from m to p.

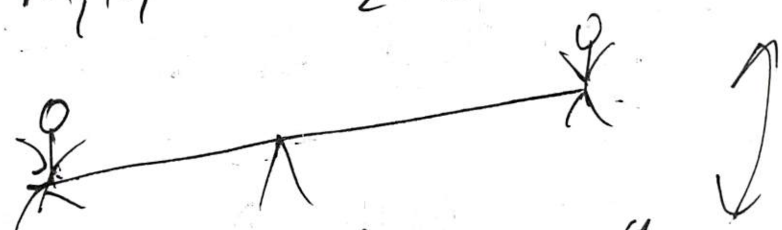
mx





condition for balancing is moments are the same

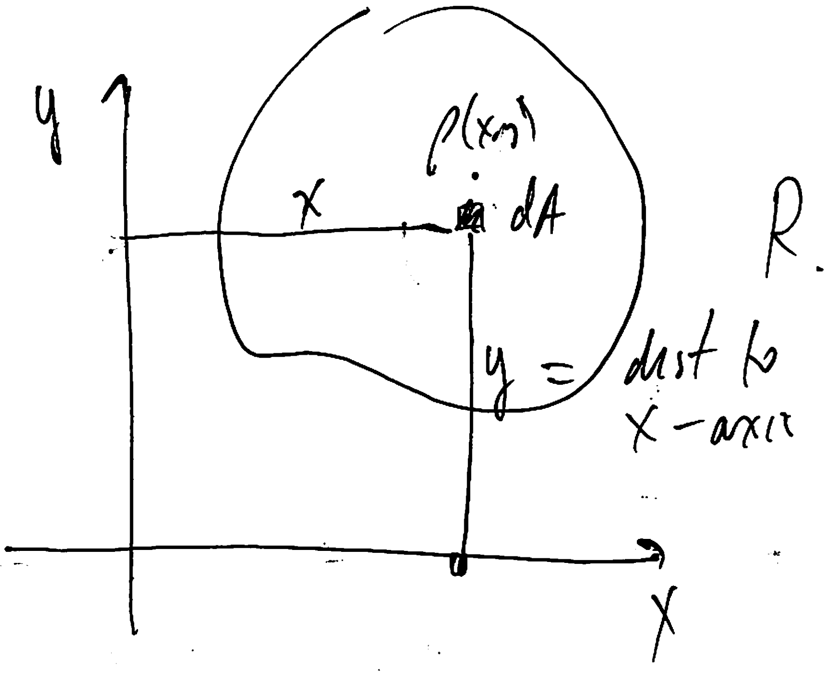
$$m_1 x_1 = m_2 x_2$$



generalizes to more than one object sides:

For balancing need

Total moment left = Total moment on the right.



M_x = moment w.r.t x axis $= \iint_R y \cdot \rho \, dA$

M_y = moment w.r.t. y-axis $= \iint_R x \rho \, dA$

Ctr of mass is denoted $(\bar{x}, \bar{y})$

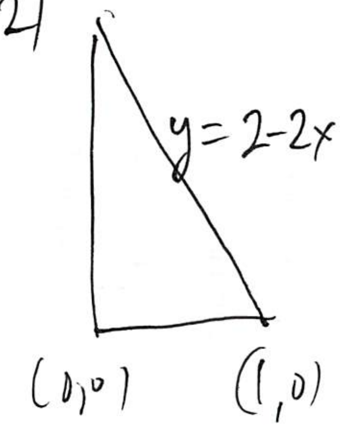
$\bar{x} = M_y / M$ M = total mass.

$\bar{y} = M_x / M$

$\bar{x} = \iint_R x \rho \, dA \Big/ \iint_R \rho \, dA$

$\bar{y} = \iint_R y \rho \, dA \Big/ \iint_R \rho \, dA$

e.g.



$\rho = 1 + 3x + y.$

$0 \le x \le 1$

$0 \le y \le 2-2x$

from before: $M = \frac{8}{3}.$

$$\bar{x} = \iint_R x\rho \, dA \Big/ \frac{8}{3}$$

$$= \frac{3}{8}\int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx$$

$$= \text{sim.} = \frac{3}{8}$$

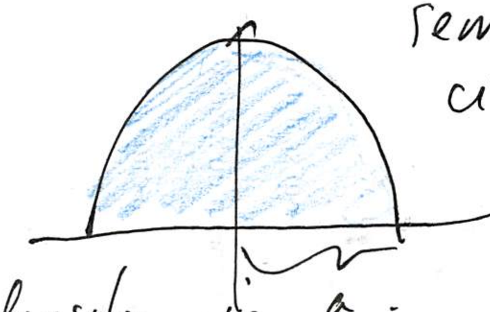
$$\bar{y} = \iint_R y\rho \, dA \Big/ \frac{8}{3}$$

$$= \frac{3}{8}\int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx$$

$$= \dots = \frac{11}{6}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{8}, \frac{11}{6}\right).$$

e.g. $R =$



semi-circle.

assume density is proportional to distance to origin.

$(\bar{x}, \bar{y})$? $\rho = K\sqrt{x^2+y^2}$

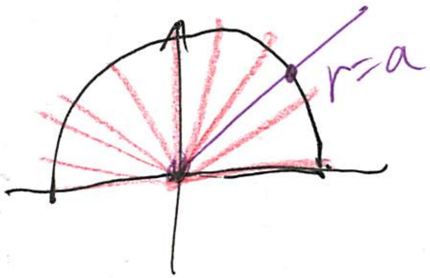
K const.

8

use polar coords.

$dA = r\,dr\,d\theta$

$\rho = K\sqrt{x^2+y^2} = Kr$



if $r \geq 0$.

$0 \leq \theta \leq \pi$

$0 \leq r \leq a$

M = mass =

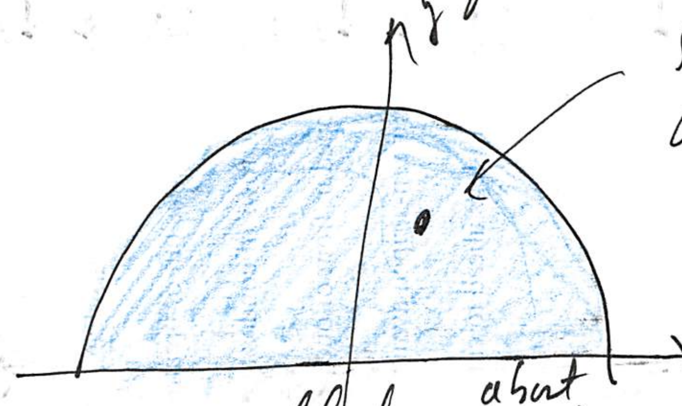
$$\int_0^{\pi}\int_0^{a} Kr \cdot r\,dr\,d\theta$$

$$= K\int_0^{\pi} \tfrac{1}{3}r^3\Big|_0^a\, d\theta = \frac{Ka^3}{3}\int_0^{\pi} d\theta$$

$$= K\pi a^3/3$$

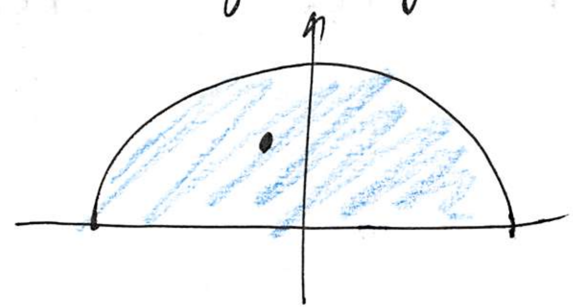
need $\bar{x}, \bar{y}$.

Claim $\bar{x} = 0$ by symmetry of the region and the density function.



suppose $\bar{x} \neq 0$ and that's the center of mass

can reflect about y axis

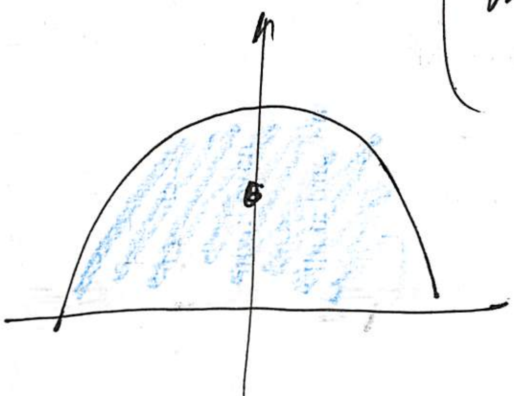


same problem but answer is now

can't get different answer because input / data didn't change.

$\Rightarrow$ ~~m~~ answer must be invariant under this symmetry

$\Rightarrow$ $\bar{x} = 0$ (c. of m must be on y-axis)



$$\iint_R x\rho\, dA = 0$$

So only need $\bar{y}$.

$$M_x = \int_0^{\pi}\int_0^{a} y\, \rho\, dA$$

$$= \int_0^{\pi}\int_0^{a} (r\sin\theta)\, Kr\; r\, dr\, d\theta$$

$$= K\int_0^{\pi}\int_0^{a} r^3 \sin\theta\, dr\, d\theta$$

$$= K\int_0^{\pi} \frac{a^4}{4} \sin\theta\, d\theta$$

$$= \frac{Ka^4}{4}\int_0^{\pi} \sin\theta\, d\theta$$

$$= \frac{Ka^4}{4}(-\cos\theta)\Big|_0^{\pi} = \boxed{Ka^4/2}$$

$$\bar{y} = \frac{M_x}{M} = \frac{Ka^4/2}{K\pi a^3/3}$$

$$= \frac{3a}{2\pi}$$

$\Rightarrow$ ans: $\left(0, \frac{3a}{2\pi}\right)$

Surface ~~integrals~~ area (15.5)

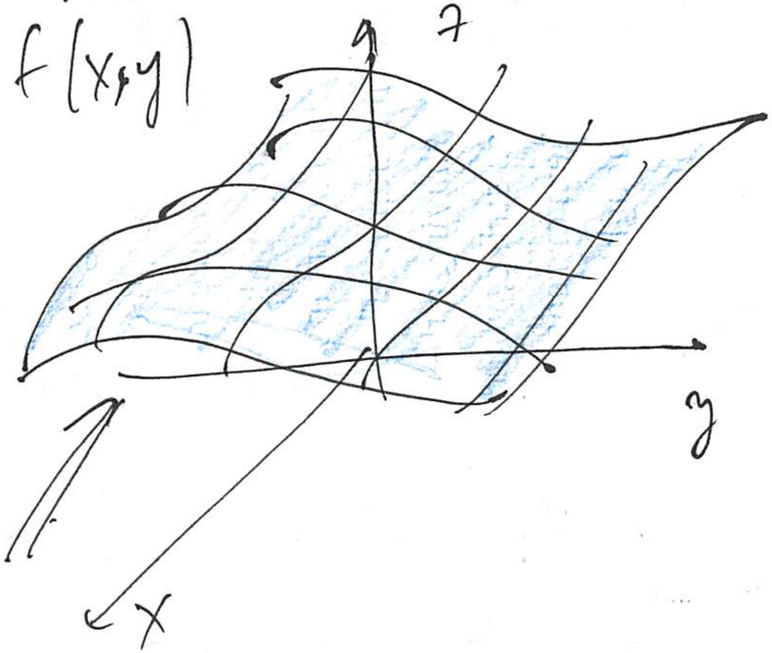
analogy: arc length 1 var

$\Updownarrow$

surface ~~integrals~~ area more than 1 var.

Basic problem: want area of a portion of the graph.

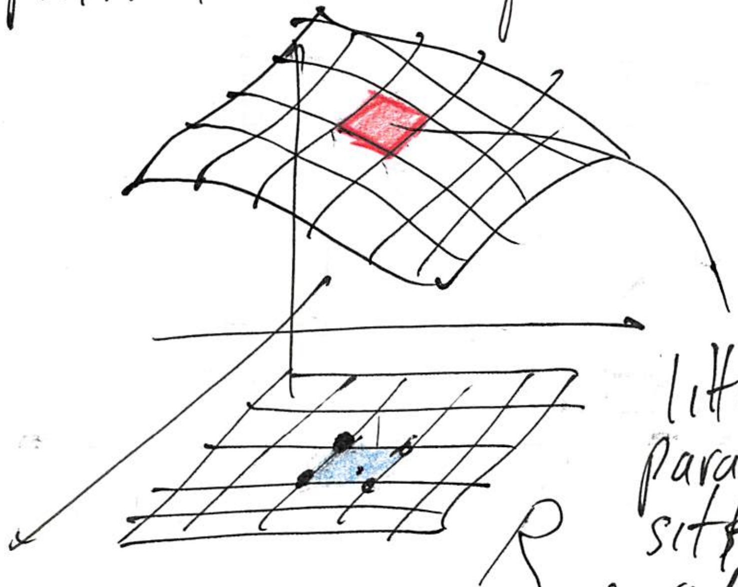
$z = f(x,y)$



want the area of this shape

Idea: chop into pieces whose area is easy to compute.

use parallelograms to approximate the surface



little para. sitting a rectangle in R.

ΔS = area of parallelogram.

want to sum these areas over the region

$$\sum\sum \Delta S$$

take limit and get

$$\iint_R dS$$

dS = tiny piece of area on graph of $f(x,y)$

to compute, we'll have an integral that looks like

$$\iint_R (\qquad) \, dA$$

something to do with f.

i.e. $dS = (\ldots) \, dA$

Answer:

$$dS = \sqrt{1+(f_x)^2+(f_y)^2}\, dA.$$

tiny area on curved shape

"distortion factor"

tiny area in base (flat)

$$\iint_R \sqrt{1+(f_x)^2+(f_y)^2}\, dA$$

= surf area.