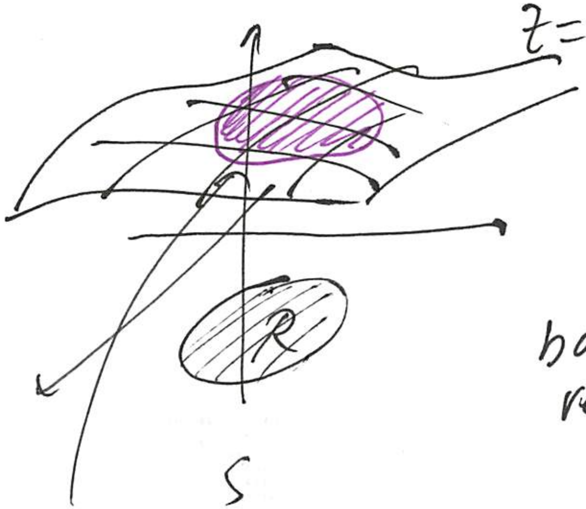
## Surface area

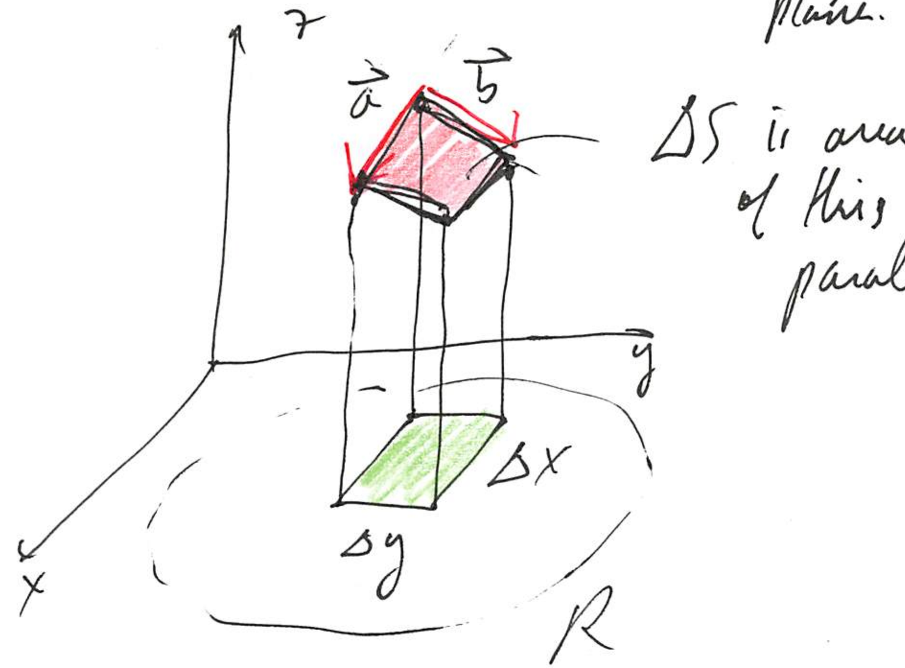



want area of this region on the graph.

Claim:

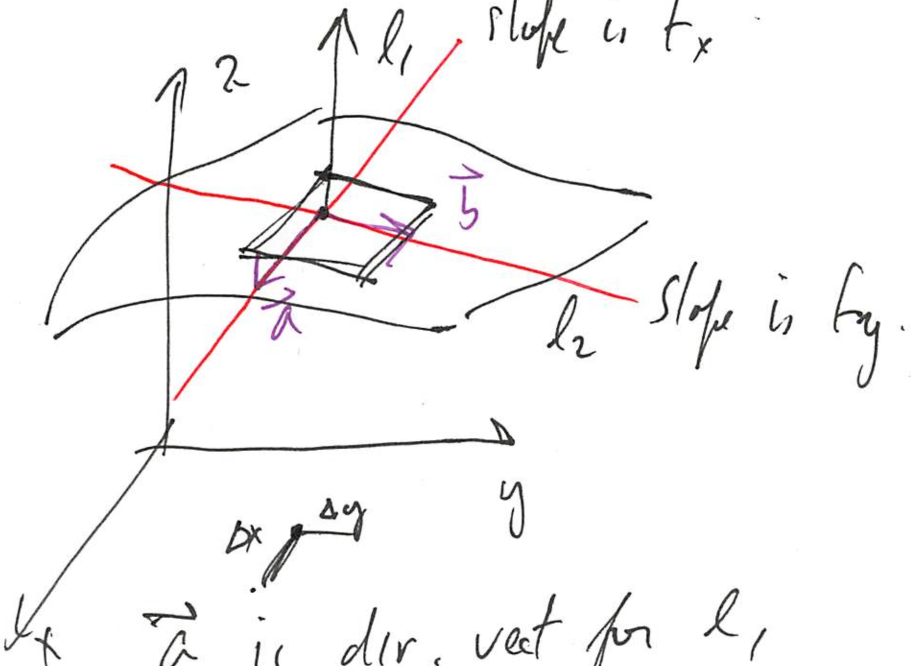
$$\iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA = S$$

To see this, model surface using parallelogram and tangent plane.




$$\Delta S = |\vec{a} \times \vec{b}|$$

Recall picture connecting tangent plane, partial derivatives to slopes of certain line




$\vec{a}$ is dir. vect for $l_1$

$\vec{b}$ . . . . . . . . $l_2$.

$$\Longrightarrow \quad \vec{a} = \langle \Delta x, 0, \Delta x f_x \rangle$$
$$\vec{b} = \langle 0, \Delta y, \Delta y f_y \rangle$$

$$\Delta S = |\vec{a} \times \vec{b}|$$

$$\vec{a} \times \vec{b} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & \Delta x f_x \\ 0 & \Delta y & \Delta y f_y \end{pmatrix}$$

$$= \Delta x \Delta y \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{pmatrix}$$

$$= \Delta x \Delta y \underbrace{\langle -f_x, -f_y, 1 \rangle}_{\perp \text{ to tangent plane.}}$$

$$|\vec{a} \times \vec{b}| = \qquad = \Delta A$$

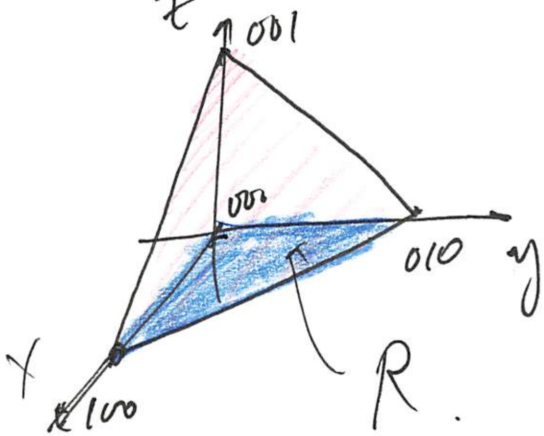
$$\Delta S = \sqrt{1 + f_x^2 + f_y^2}\, \Delta x \Delta y.$$

$$\boxed{\iint_R dS = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA}.$$

compare with arc length from calc 2:

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

e.g. area of front face of a tetrahedron with verts 000, 100, 010, 001




need front face as

$z = f(x,y)$

plane is $x+y+z=1$

$\Longrightarrow$ $z = 1-x-y. = f$

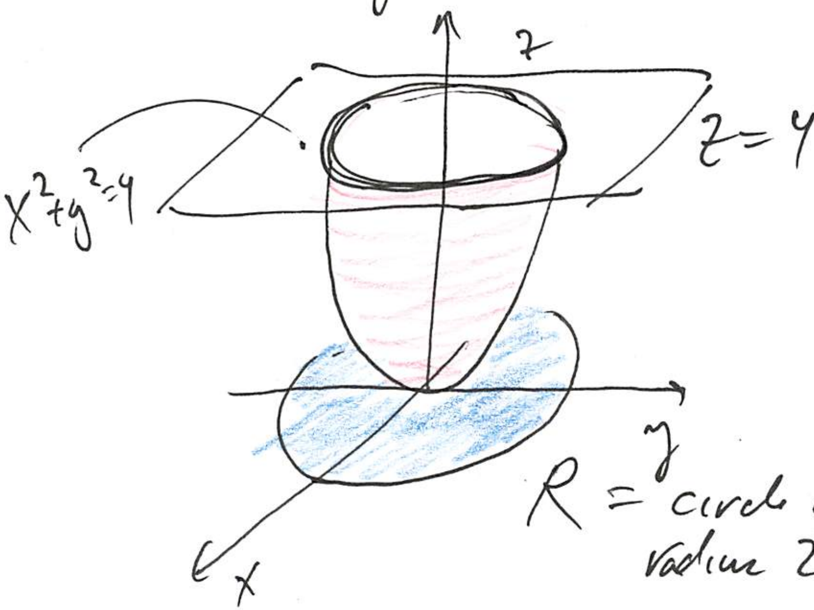
$$\sqrt{1+f_x^2+f_y^2} = \sqrt{1+(-1)^2+(-1)^2} = \sqrt{3}$$

$$S = \iint_R \sqrt{3}\, dA = \sqrt{3}\iint_R dA$$

$$= \sqrt{3}\,\text{area}(R)$$

$$= \boxed{\frac{\sqrt{3}}{2}}.$$

e.g. surf area of paraboloid $z = x^2 + y^2$ under $z = 4$




$R$ = circle of radius 2.

$$\iint_R dS \quad \text{polar probably.}$$

$$\sqrt{1 + f_x^2 + f_y^2} \qquad f_x = 2x \quad f_y = 2y$$

$$= \sqrt{1 + 4x^2 + 4y^2}$$

use polar

$$\iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA \qquad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

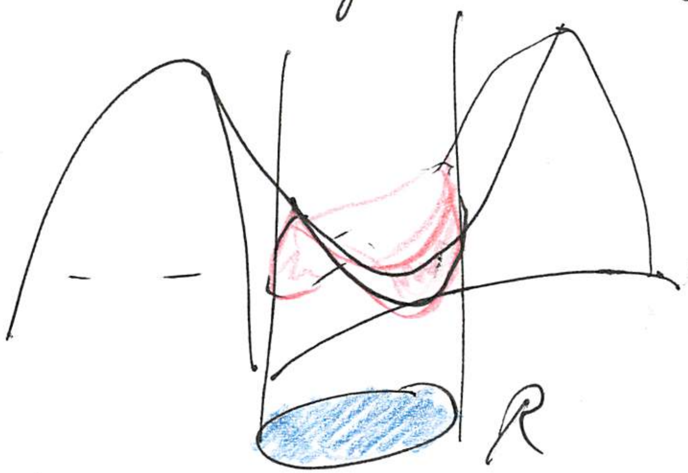
$$= \frac{1}{8} \int_0^{2\pi} \int_1^{17} u^{1/2}\, du\, d\theta \qquad u = 1 + 4r^2, \quad \frac{1}{8} du = \cancel{8} r\, dr$$

$$= \frac{1}{8} \int_0^{2\pi} \frac{2}{3} u^{3/2} \Big|_1^{17} d\theta$$

$$= \frac{1}{12} \left(17^{3/2} - 1\right) \int_0^{2\pi} d\theta$$

$$= \frac{2\pi}{12}\left(17^{3/2} - 1\right)$$

e.g. area of $z = xy$ inside the cylinder $x^2+y^2 \le 1$




in $xy$ plane: $R$ = unit disk.

$x^2+y^2 \le 1$

$f = xy$

$f_x = y$, $f_y = x$

$$\sqrt{1 + \underbrace{y^2+x^2}_{r^2}}\, dA.$$

use polar.

$$\int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\; r\,dr\,d\theta$$

$u = 1+r^2$

$\frac{1}{2} du = r\,dr$

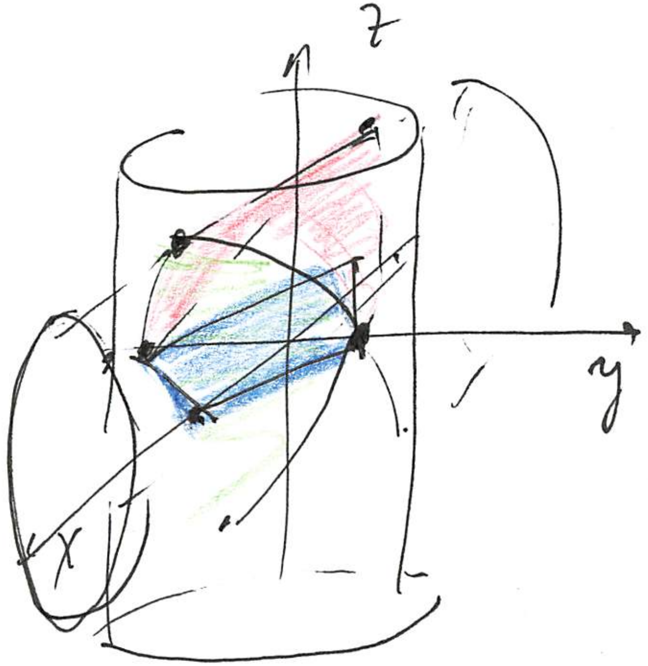
$$\frac{1}{2}\int_0^{2\pi}\int_1^2 u^{1/2}\,du\,d\theta$$

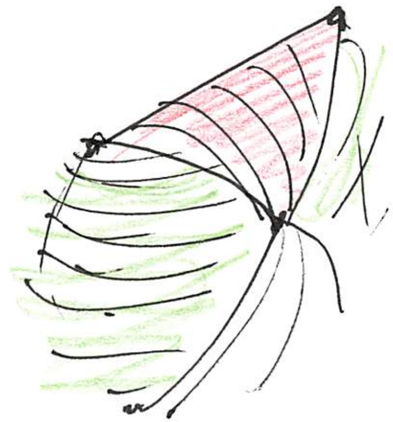
$$= \frac{1}{3}\left(2^{3/2}-1\right)\int_0^{2\pi} d\theta$$

$$= \boxed{2\pi/3\left(2^{3/2}-1\right)}$$

e.g. Find surface area of the intersection of 2 solid cylinders radius 1 meeting at 90°.

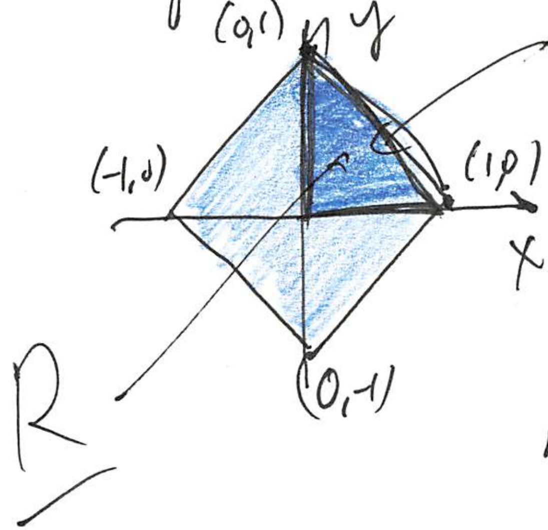
can take $x^2+y^2 \leq 1$
$y^2+z^2 \leq 1$.



need 4x the area of any single face

red face sits over




can take 4x area over this to get red area.

upper red face comes from

$y^2+z^2=1$

$z=\sqrt{1-y^2}=f.$

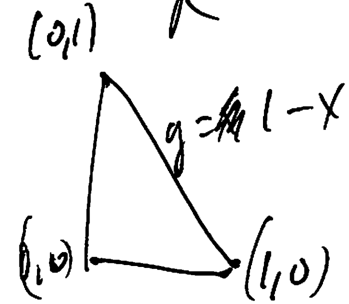
$$\Rightarrow 16\iint_R \sqrt{1-f_x^2-f_y^2}\,dA$$

$f_x=0$

$f_y=y/\sqrt{1-y^2}$

$$16\iint_R \sqrt{1+\frac{y^2}{1-y^2}}\,dA$$

$$=16\iint_R \sqrt{\frac{1}{1-y^2}}\,dA$$




$0\le x\le 1$

$0\le y\le 1-x$

$$=16\int_0^1\int_0^{1-x}\sqrt{\frac{1}{1-y^2}}\,dy\,dx$$

at least we set it up...

## Triple integrals

want to generalize double integrals to functions of 3 variables

$\int_a^b f(x)\,dx$ 1D

$dx$ = tiny element of length.

$\iint_R f(x,y)\,dA$ 2D double

$dA$ tiny element of area in base region $R$.

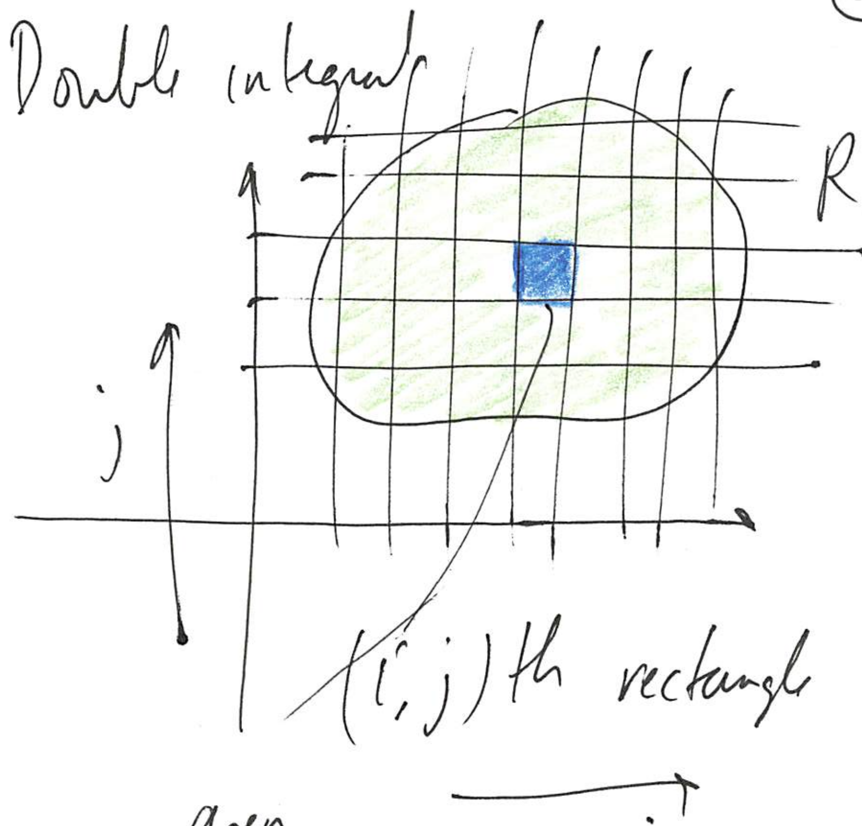
Triple integral 3D

$\iiint_E f(x,y,z)\,dV$

$dV$ tiny element of volume

$E \leftarrow$ 3 dimensional solid region

Double integrals




area $\Delta A$.

Let $(x_{ij}^*, y_{ij}^*)$ be sample pt in $i,j$ th rectangle

$$\sum_i \sum_j f(x_{ij}^*, y_{ij}^*)\, \Delta A.$$

Take limit as $\Delta A \to 0$

and get

$$\iint_R f(x,y)\, dA$$

$\Delta A = \Delta x \Delta y$

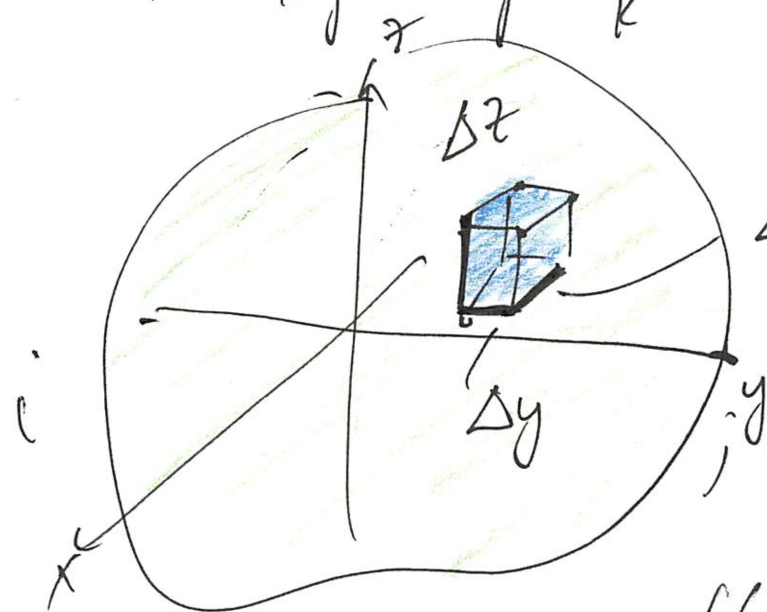
$dA = dx\, dy$



Triple integrals

$E$ 3D region

in $xyz$ space




$f(x,y,z)$

$\Delta V = \Delta x \Delta y \Delta z$

boxes indexed by $i, j, k$.

$(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$

sample point in box $(i,j,k)$

plug into $f$, then sum over all $i, j, k$.

$$\sum_i \sum_j \sum_k f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$$

take limit $\Delta V \to 0$

$$\boxed{\iiint_E f(x,y,z)\, dV}$$

Tricky aspects:

1. $E$ is now 3D. more difficult to handle.

2. use technique of iterated integration

Double: $\displaystyle \underbrace{\int \Big[ \underbrace{\int f \, dy}_{y \text{ first}} \Big] dx}_{x}$

$dy\,dx$

also have possibility of $dA = dx\,dy$

now have more orders of integration : 6

$dx\,dy\,dz$ $dy\,dx\,dz$

$dy\,dz\,dx$

$dx\,dz\,dy$

$dz\,dy\,dx$

$dz\,dx\,dy$

3 Double : $xy$
polar

Triple : $xyz$
cylindrical
spherical