last time: Triple integrals

E region in 3D

$dV = dx\,dy\,dz$

$$\iiint_E f(x,y,z)\,dV$$

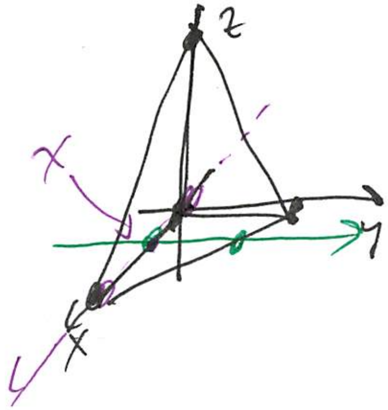
do as an iterated integral

e.g.

$$\underbrace{\int}_{\text{numbers}} \underbrace{\int}_{\text{fns of } z \text{ are OK}} \underbrace{\int}_{\text{fns of } y \text{ and } z \text{ OK}} f \; \overbrace{dx}^{x} \; \overbrace{dy}^{y} \; \overbrace{dz}^{z}$$

e.g. Volume of a tetrahedron

verts at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

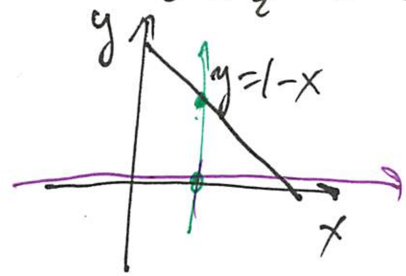


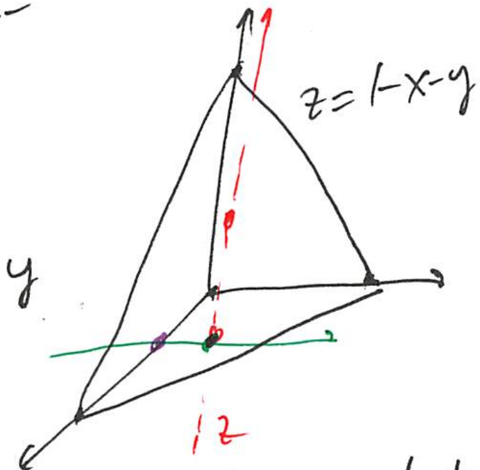
$$\iiint_E dV = \mathrm{vol}(E).$$

$0 \le x \le 1$

$0 \le y \le 1-x$

$0 \le z \le 1-x-y$





order of integration

$dV = dz\,dy\,dx$

$$\int_0^1 \int_0^{1-x} \left(\int_0^{1-x-y} dz \right) dy\, dx$$

$$= \int_0^1 \left(\int_0^{1-x} 1-x-y \; dy \right) dx$$

$$= \int_0^1 y - xy - \tfrac{1}{2}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 (1-x) - x(1-x) - \tfrac{1}{2}(1-x)^2 \, dx$$

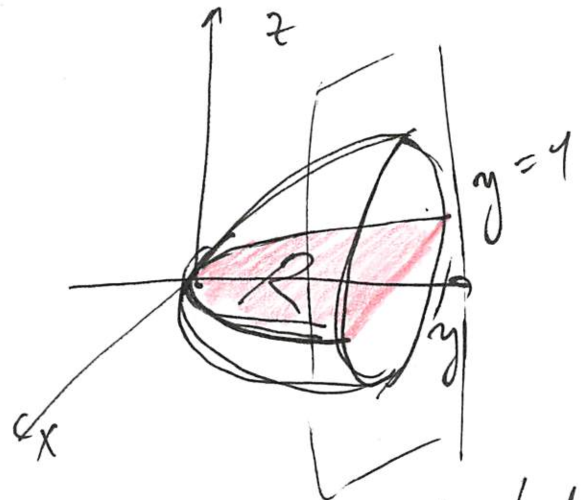
$$= \int_0^1 1-x -x+x^2 - \tfrac{1}{2} + x - \tfrac{1}{2}x^2 \; dx$$

$$= \int_0^1 \tfrac{1}{2} - x + \tfrac{1}{2}x^2 \; dx$$

$$= \frac{x}{2} - \frac{x^2}{2} + \frac{1}{6}x^3 \Big|_0^1 = \boxed{\frac{1}{6}}$$

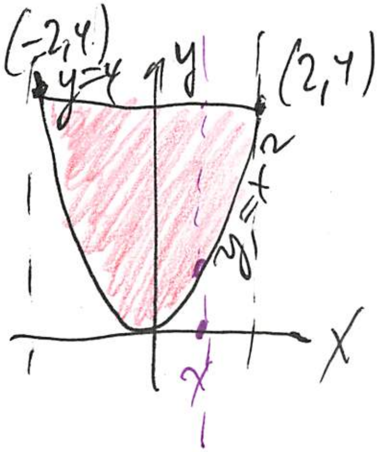
e.g. $\iiint_E \sqrt{x^2+z^2}\, dV$

E is bounded by $y = x^2+z^2$, $y=4$

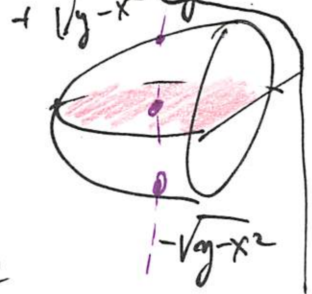


Method 1: use red shape as a "base region" R for E

First describe R using x, y, then get E by incorporating z.



$$R: \quad -2 \le x \le 2$$
$$x^2 \le y \le 4$$
$$-\sqrt{y-x^2} \le z \le \sqrt{y-x^2}$$

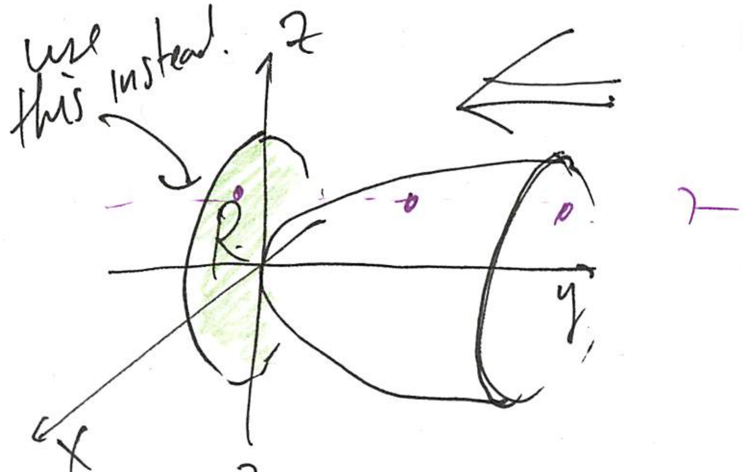


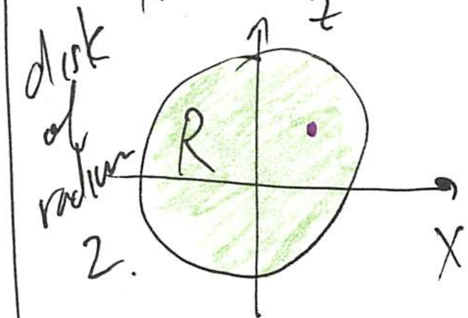
$$y = x^2 + z^2$$

$$\Rightarrow z = \pm\sqrt{y-x^2}$$

$$\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\; dz\, dy\, dx$$

Not easiest way to do it.
Better way: use diffent base in xz plane





now use polar to describe R,
use xz instead of xy

$$x = r\cos\theta$$
$$z = r\sin\theta$$

$$R: \begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$$

$$y = x^2 + z^2 = r^2 \qquad r^2 \le y \le 4$$

4

$f(x,y,z) = \sqrt{x^2+z^2} = r.$

$dV = dy\, dx\, dz$

$= dy\, (r\, dr\, d\theta)$

$$\Rightarrow \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r \cdot r \cdot dy\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r^2 \int_{r^2}^{4} dy\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r^2(4-r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (4r^2-r^4)\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{4}{3}r^3 - \frac{r^5}{5}\Big|_0^2\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{32}{3} - \frac{32}{5}\right) d\theta$$

$$= 32\left(\frac{1}{3}-\frac{1}{5}\right) 2\pi$$

$$= \boxed{\frac{128\pi}{15}}$$

Cylindrical / Spherical coordinates and triple integrals.

analogues of polar coords for 3D.

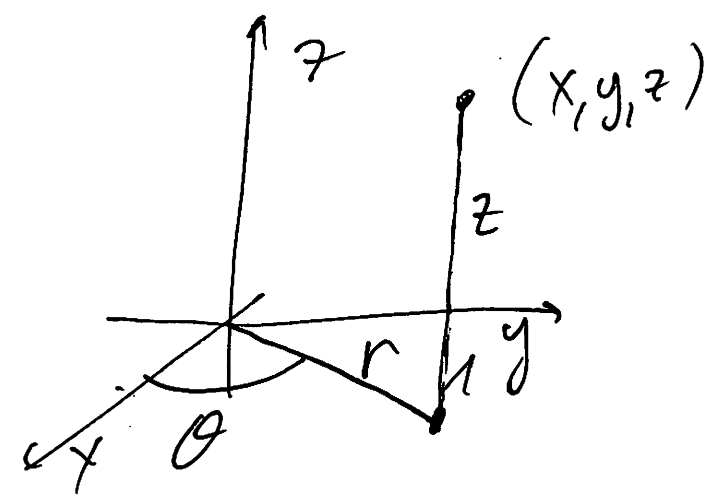
2D	x, y	2 lengths
	r, θ	1 length, 1 angle
3D	x, y, z	3 lengths
cylindrical coords	r, θ, z	2 lengths, 1 angle
spherical coords	ρ, θ, φ	1 length, 2 angles

2D : $dA = dx\,dy$
$= r\,dr\,d\theta$

(1) what are these new coordinates, and how are they related to x, y, z ?

(2) $dV = ?$ in these coords.

Cylindrical: do polar in x, y plane; keep z the same



Observe: same r, same θ for usual polar coordinates

$x = r\cos\theta$

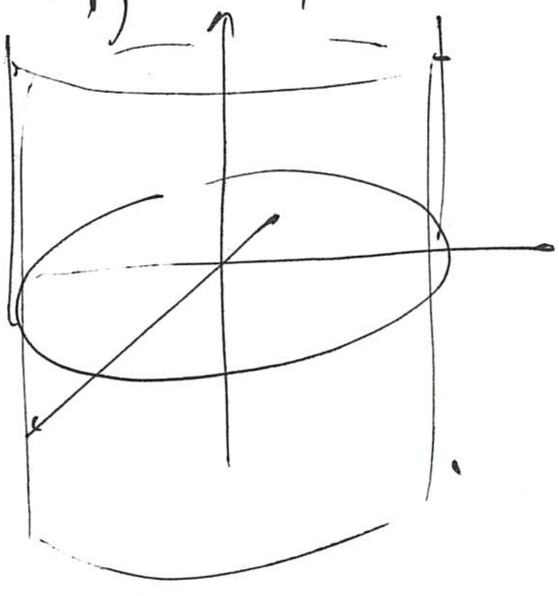
$y = r\sin\theta$

$z = z$

why "cylindrical"?

Fix r, ~~Fix z~~ z be free, $0 \le \theta \le 2\pi$

get cylinder.



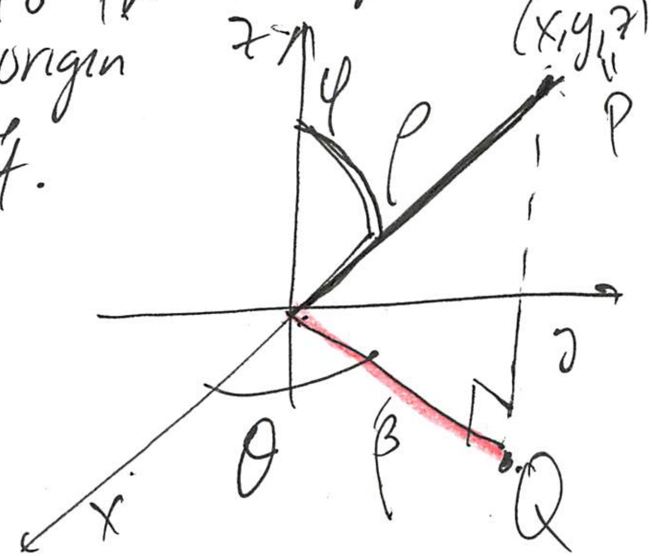
useful for problems with rotational symmetry about z-axis.

Spherical coords:

ρ = distance from pt to origin

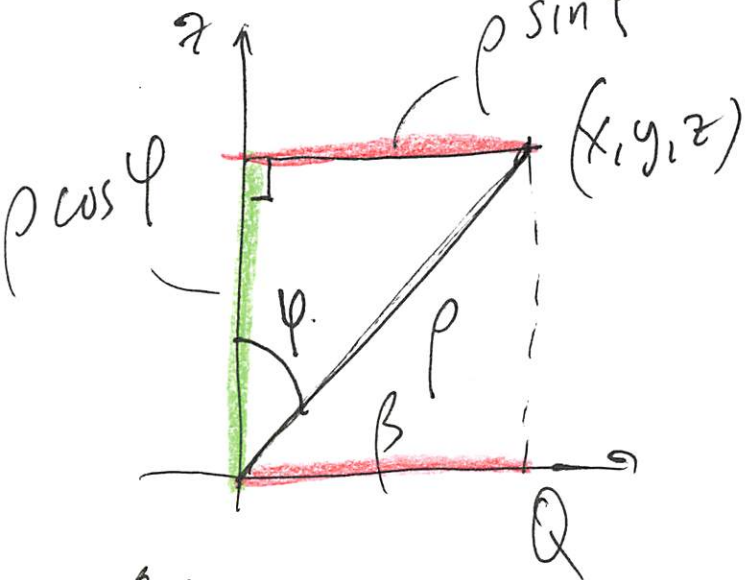
θ = same as θ in cylindrical.

φ = angle from positive z-axis to the line segment from origin to the point.



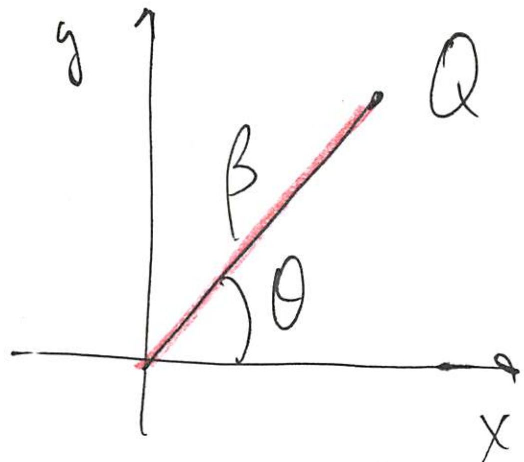
$$\boxed{\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \end{aligned}}$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$



~~xyz~~

in xy plane:



$$x = \beta \cos\theta$$

$$y = \beta \sin\theta$$

and $\beta = \rho \sin\varphi$

next time: dV in these coords examples.