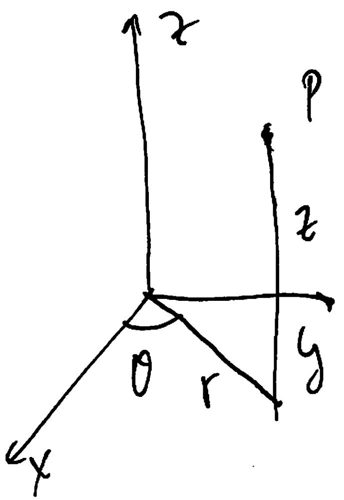
Last time: cylindrical
spherical

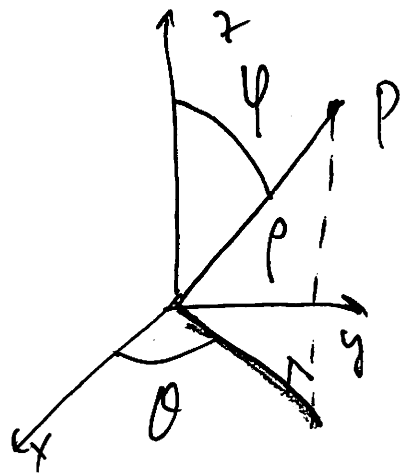
Cylin



r, θ, z

$r \geq 0$
$0 \leq \theta \leq 2\pi$
any z

Spher.



ρ, θ, φ

$\rho \geq 0$
$0 \leq \theta \leq 2\pi$
$0 \leq \varphi \leq \pi$

Conversion formulas

Cyl.

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

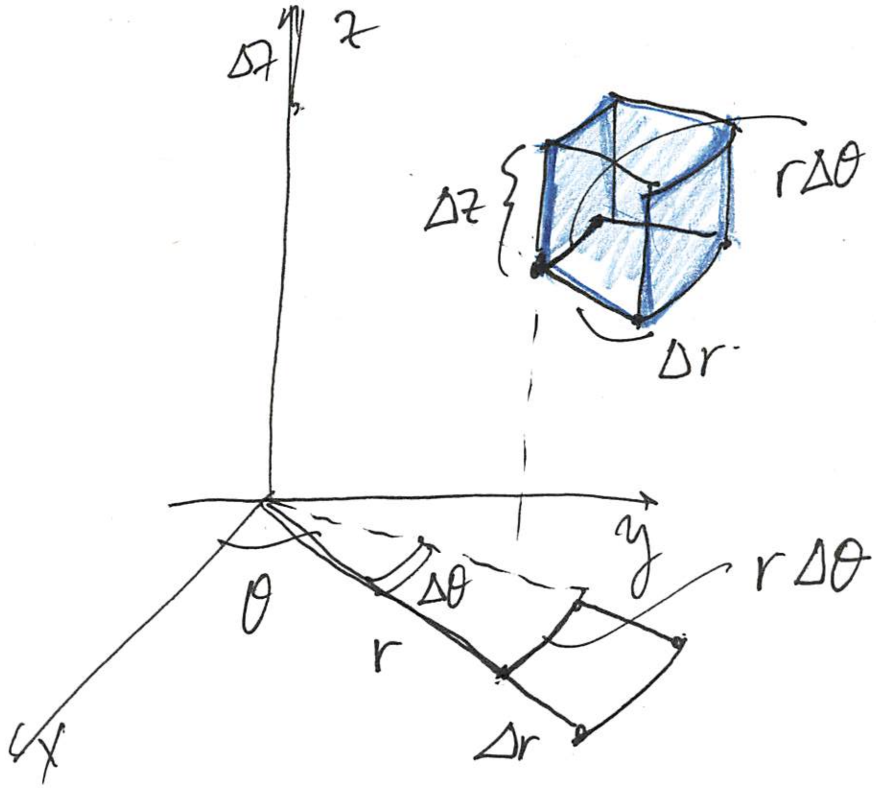
Sph.

$x = \underline{\rho\sin\varphi}\cos\theta$
$y = \underline{\rho\sin\varphi}\sin\theta$
$z = \rho\cos\varphi$

Triple ints in cylindrical coordinates

$$\iiint_E f \, \underline{dV}$$

$dV = dx\,dy\,dz$



$r \to r+\Delta r$

$\theta \to \theta + \Delta\theta$

$z \to z + \Delta z$

Vol of box

$\approx$

$$\Delta V = r\,\Delta r\,\Delta\theta\,\Delta z$$

take limit: $dV = r\,dr\,d\theta\,dz$

$$dx\,dy\,dz$$

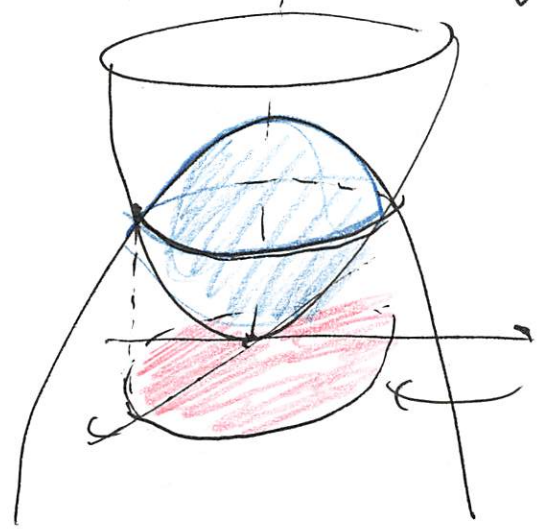
$$\Downarrow$$

$$r\,dr\,d\theta\,dz$$

e.g. compute volume between

$z = x^2 + y^2$

$z = 5 - x^2 - y^2$



conceive as a base region

To find radius,

$$x^2+y^2 = 5-x^2-y^2$$

$$5 = 2x^2+2y^2$$

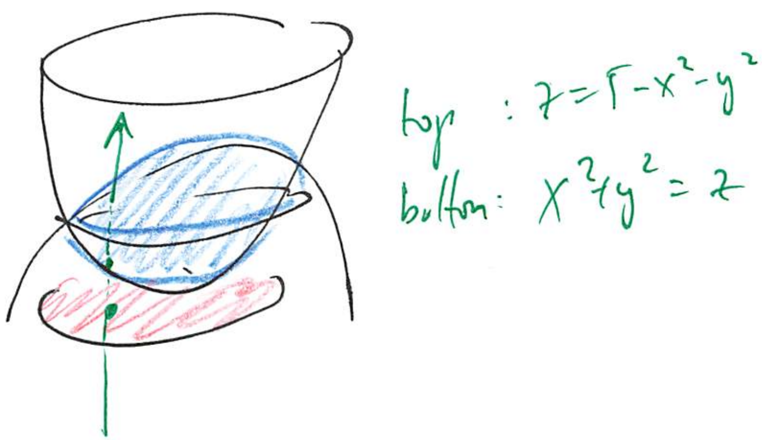
$$x^2+y^2 = \frac{5}{2}$$

radius is $\sqrt{\frac{5}{2}}$.

Base region $\quad 0 \le \theta \le 2\pi$

$$0 \le r \le \sqrt{\frac{5}{2}}$$

rest: $\quad ? \le z \le ?$



$$0 \le \theta \le 2\pi$$

$$0 \le r \le \sqrt{\frac{5}{2}}$$

$$x^2+y^2 \le z \le 5-x^2-y^2$$

$$\Rightarrow \quad r^2 \le z \le 5-r^2$$

$$\int_0^{2\pi}\int_0^{\sqrt{\frac{5}{2}}}\int_{r^2}^{5-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{\frac{5}{2}}} (5-2r^2)r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{\frac{5}{2}}} (5r-2r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{5}{2}r^2 - \frac{1}{\cancel{4}2}r^4 \Big|_0^{\sqrt{\frac{5}{2}}} d\theta$$

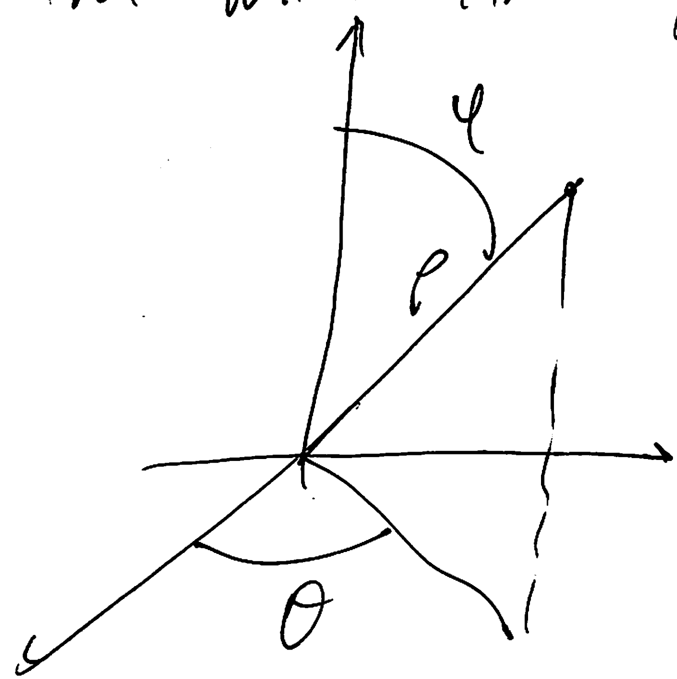
$$= \int_0^{2\pi} \left(\frac{25}{4} - \frac{1}{\cancel{4}2} \cdot \frac{25}{4} \right) d\theta$$

$$= \frac{\cancel{3}1}{\cancel{4}2} \cdot \frac{25}{4} \int_0^{2\pi} d\theta =$$

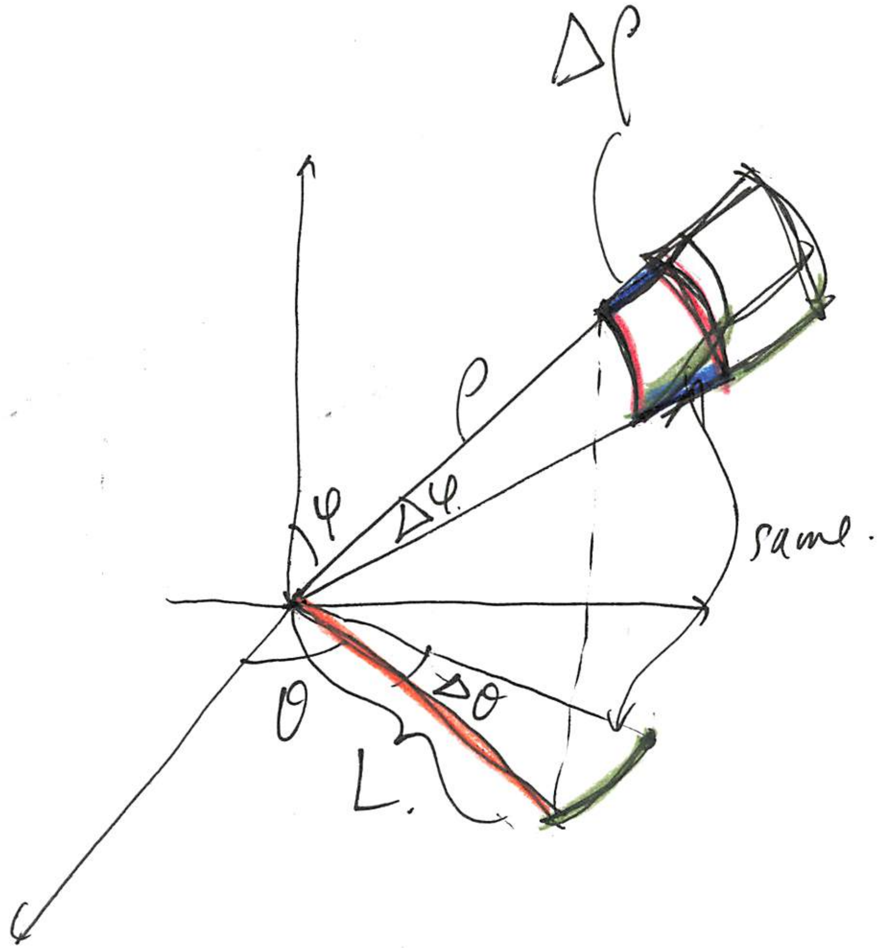
$$= \cancel{\boxed{\frac{75\pi}{8}}} \quad \boxed{\frac{25}{4}\pi}.$$

<u>Spherical triple integrals</u> (4)

need dV. will involve $d\rho$, $d\theta$, $d\varphi$, but there will be extra stuff.



$\Delta\rho$
$\Delta\theta$
$\Delta\varphi$



$$\Delta V = \underline{\Delta\rho} \cdot \underline{(\rho\Delta\varphi)} \cdot \underline{(L\,\Delta\theta)}$$

from conversion derivation,

$$L = \rho \sin\varphi$$

$$\Rightarrow \Delta V = \Delta\rho \cdot \rho\,\Delta\varphi \cdot \rho\sin\varphi\,\Delta\theta$$

take limit:

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

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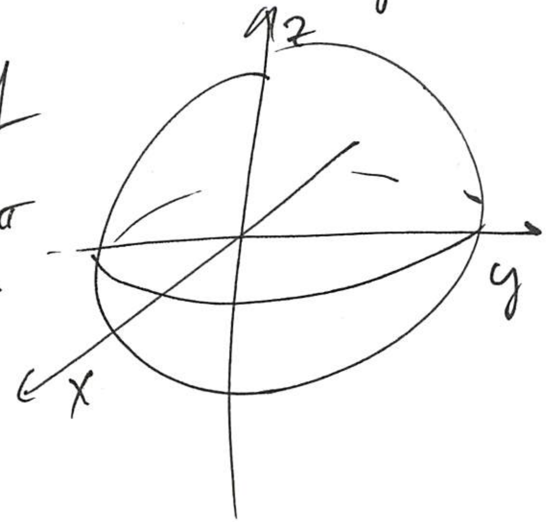
e.g. Volume of unit sphere

$0 \le \rho \le 1$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

E



$$\iiint_E dV = \text{vol (sphere)}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \tfrac{1}{3}\rho^3\Big|_0^1 \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi} \, d\theta$$

$$= \frac{2}{3}\int_0^{2\pi} d\theta = \boxed{\frac{4\pi}{3}}$$

$$\boxed{\frac{4}{3}\pi R^3}$$

Remark: in many problems, θ, φ are not part of f and limits are $0 \le \theta \le 2\pi$ and ρ independent of θ, φ. $0 \le \varphi \le \pi$ can use the "precomputation"

$$\int_0^{2\pi}\int_0^{\pi} \sin\varphi \, d\varphi \, d\theta = 4\pi$$

e.g. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\left(\int_0^1 \rho^2 \, d\rho\right) \sin\varphi \, d\varphi \, d\theta$

$$= \left(\int_0^1 \rho^2 d\rho\right)\left(\underbrace{\int_0^{2\pi}\int_0^{\pi} \sin\varphi \, d\varphi \, d\theta}_{4\pi}\right)$$

$$= \frac{1}{3}\cdot 4\pi$$

$$= 4\pi/3$$

e.g. E solid ball of radius 1 @ origin

$$f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$$

$$\iiint_E f\,dV = ?$$

$$\rho^2 = x^2+y^2+z^2$$

so $f = e^{\rho^3}$

$$\iiint_E e^{\rho^3}\rho^2 \sin\varphi \, d\varphi\, d\theta\, d\rho$$

$0 \le \rho \le 1$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^{\rho^3}\rho^2 \sin\varphi\, d\varphi\, d\theta\, d\rho$$

$$= \left(\int_0^1 e^{\rho^3}\rho^2\, d\rho\right)\underbrace{\left(\int_0^{2\pi}\int_0^{\pi} \sin\varphi\, d\varphi\, d\theta\right)}_{4\pi}$$

$u = e^{\rho^3}$

$\frac{1}{3}du = e^{\rho^3}\rho^2\, d\rho$

$0 \le \rho \le 1 \Longrightarrow 1 \le u \le e$

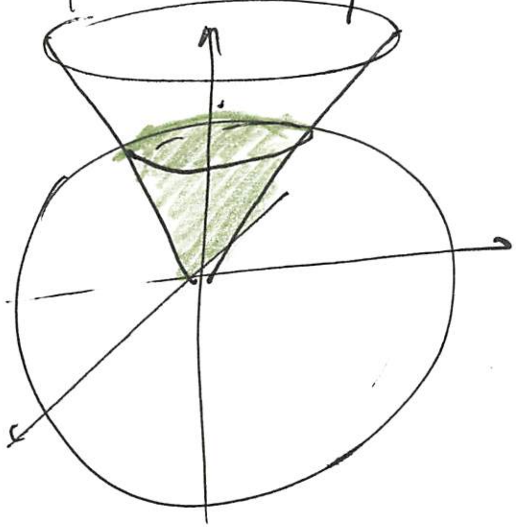
$$= \frac{4\pi}{3}\int_1^e u\,du = \frac{4\pi}{3}\,\frac{1}{2}u^2\Big|_1^e$$

$$= \frac{2\pi}{3}\left(e^2 - 1\right)$$

e.g. find volume inside cone

$z = \sqrt{x^2+y^2}$

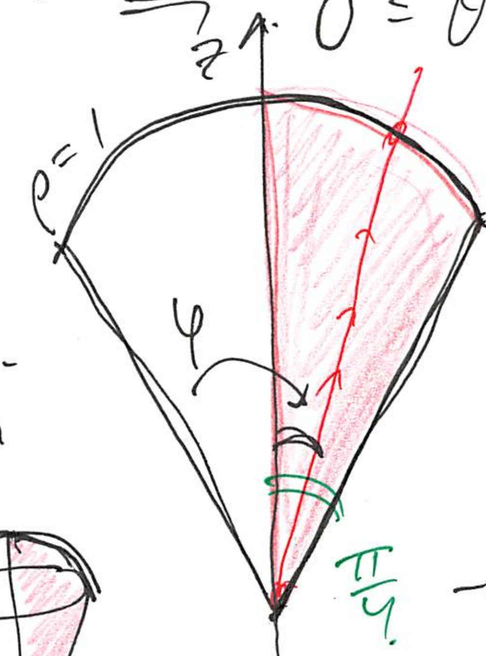
and the unit sphere.



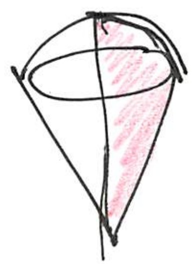
Spherical.

sym about z axis

$\Rightarrow 0 \le \theta \le 2\pi$



Cross-section



$0 \le \rho \le 1$

$-\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}$

vs

✓ $\Rightarrow 0 \le \varphi \le \frac{\pi}{4}$?

This gives the shape once after rotate red shape using θ.

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

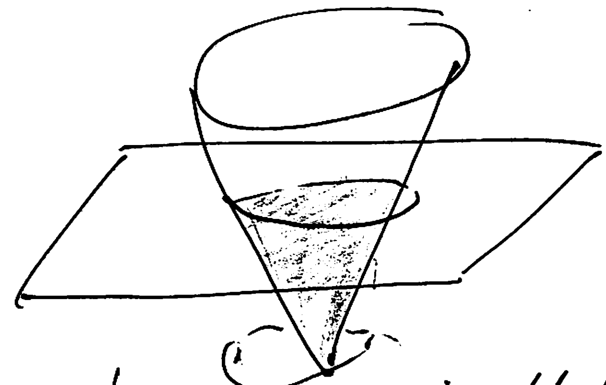
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} -\cos\varphi \Big|_0^{\frac{\pi}{4}} \, d\theta$$

$$= \frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)\int_0^{2\pi} d\theta =$$

$$\frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right).$$

Variation: use cone and plane $z=1$



because top is flat, should use cylindrical instead.

region E : $0 \le r \le 1$

$0 \le \theta \le 2\pi$

$\sqrt{x^2+y^2} \le z \le 1$

$r \le z \le 1$

Next time: – another spherical example

– Jacobian (general variable change)