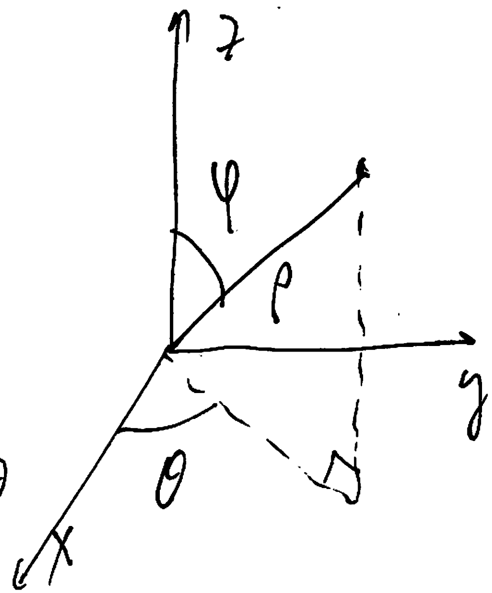
Last time: triple integrals w/ spherical coords.

ρ, θ, φ

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$



$dV =$

$\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

Example convert to spherical

$$\underbrace{\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}}}_{E \text{ region}} (x^2+y^2+z^2) \, dz \, dx \, dy$$

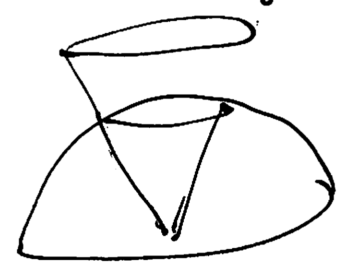
Integrand: $x^2+y^2+z^2 = \rho^2$

$$\left.\begin{array}{l} 0 \le y \le 3 \\ 0 \le x \le \sqrt{9-y^2} \end{array}\right\}$$

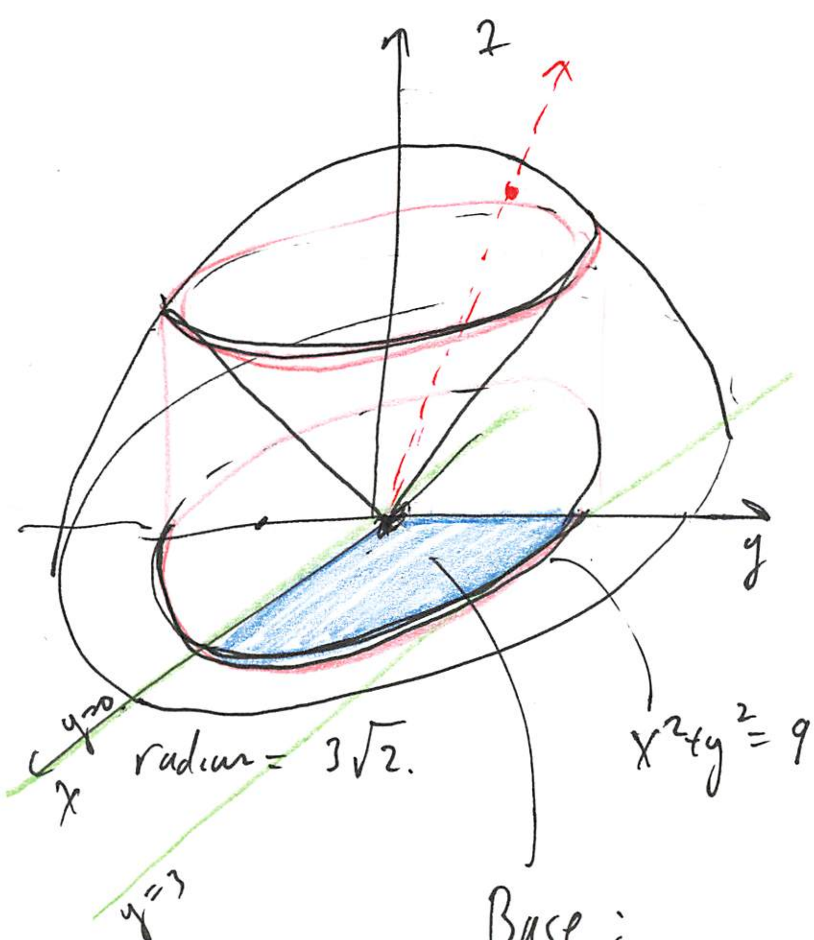
$\sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2}$

$z = \sqrt{18-x^2-y^2}$ upp hemisphere center at origin radius $= \sqrt{18} = 3\sqrt{2}$.

$z = \sqrt{x^2+y^2}$ cone, vert @ origin sides make angle $\pi/4$ w.r.t. z-axis



Base: $x = \sqrt{9-y^2}$ upp semi circle rad $= 3$



Base :

$0 \le y \le 3$

$0 \le x \le \sqrt{9-y^2}$

red circle: $\sqrt{x^2+y^2} = \sqrt{18-x^2-y^2}$

$\Leftrightarrow \; 2x^2+2y^2=18$

$\Leftrightarrow \; x^2+y^2=9 \Rightarrow$ rad $=3$

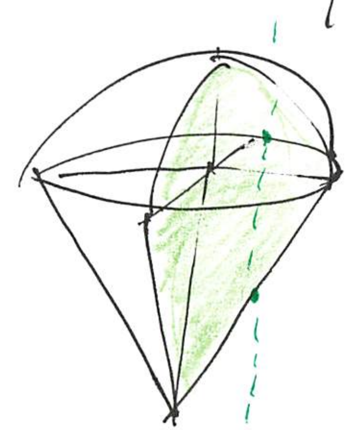
$0 \le \theta \le \frac{\pi}{2}.$

$0 \le \rho \le 3\sqrt{2}$

$0 \le \varphi \le \frac{\pi}{4}$

Ans: $\iiint \rho^2 \, dV$

$$= \int_0^{\frac{\pi}{2}} \int_0^{3\sqrt{2}} \int_0^{\pi/4} \rho^4 \sin\varphi \, d\varphi \, d\rho \, d\theta$$



Change of variables in multiple integrals / Jacobian

1 var calc: $\int_a^b f(x)\,dx$

u - substitution

$x = g(u)$ $\quad dx = g'(u)\,du$

$$\int_a^b f(x)\,dx = \int_{u_1}^{u_2} f(g(u))\,g'(u)\,du$$

solve $a = g(u_1)$

$b = g(u_2)$

note: ① backwards from usual way of thinking about u substitution; still correct.

② 3 things have to change.

- f $\quad f(g(u))$
- differential $\quad g'(u)\,du$
- limits

analogue now: go from (x, y) in a double integral to 2 new variables (u, v) simultaneous substitution

$x = \ldots u, \ldots v$

$y = \ldots u \ldots v$

$$\iint_R f(x,y)\,dxdy = \iint_{*} (*)(u,v)\ (*)$$

- $f(x,y) \Rightarrow$ something in u,v
- dA must be adjusted
- limits must be adjusted

Example: polar coordinates.

$$x,y \rightsquigarrow r,\theta$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} (*)$$

$$\iint_R f(x,y)\,dx\,dy$$

$$\|$$

$$\iint_R f(r\cos\theta, r\sin\theta)\,(\ \ )\,dr\,d\theta$$

extra.

diff description in terms of r,θ.

$$dA = dxdy$$
$$= r\,dr\,d\theta$$

additional factor coming from the coordinate change.

extra factor is extractable from $(*)$. general technique to find it:

$$\boxed{\text{Jacobian}}$$

Suppose we change coords from x, y to u, v.

function name

i.e. variable

$$x = x(u,v)$$
$$y = y(u,v)$$

$$dx\,dy = \underbrace{}_{\text{Jacobian}}\, du\,dv.$$

to compute it,

$$\left| \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \right|$$

$$= \left| x_u y_v - y_u x_v \right|$$

Notation

$$= \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

example polar

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\boxed{\begin{array}{l} u = r \\ v = \theta \end{array}}$$

$$\text{Jacobian}: \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right|$$

$$= \left| \det \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} \right|$$

$$= \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right|$$

$$= \left| r\cos^2\theta + r\sin^2\theta \right|$$

$$= |r| = r \quad \text{if take } r \geq 0$$

$$\Longrightarrow \quad dx\,dy = \underline{r}\,dr\,d\theta$$

We also must change the limits

e.g. R = region bounded by

$y = -x + 4$

$y = x + 1$

$y = \frac{x}{3} - 4/3$

do change of vars

$x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$

1. express R using u, v.
2. compute Jacobian of this change of vars.

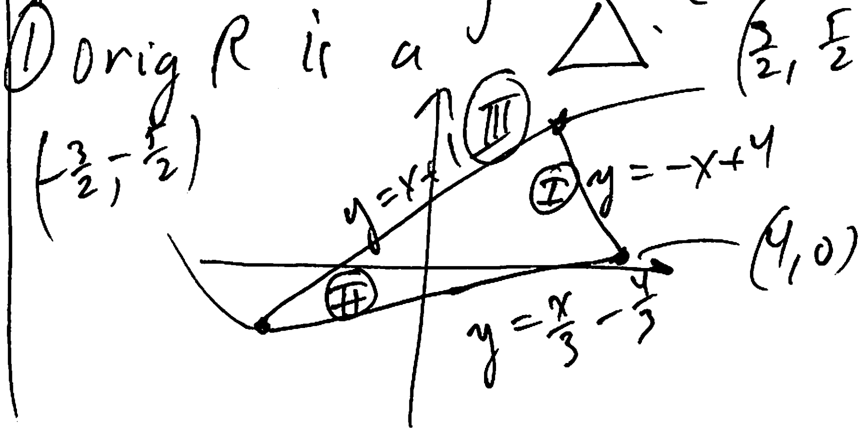
(2): $\left| \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \right|$

$$= \left| \det \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \right|$$

$$= \left| -\tfrac{1}{4} - \tfrac{1}{4} \right| = \tfrac{1}{2}$$

$$dx\,dy = \tfrac{1}{2}\,du\,dv.$$

(1) orig R is a $\triangle$



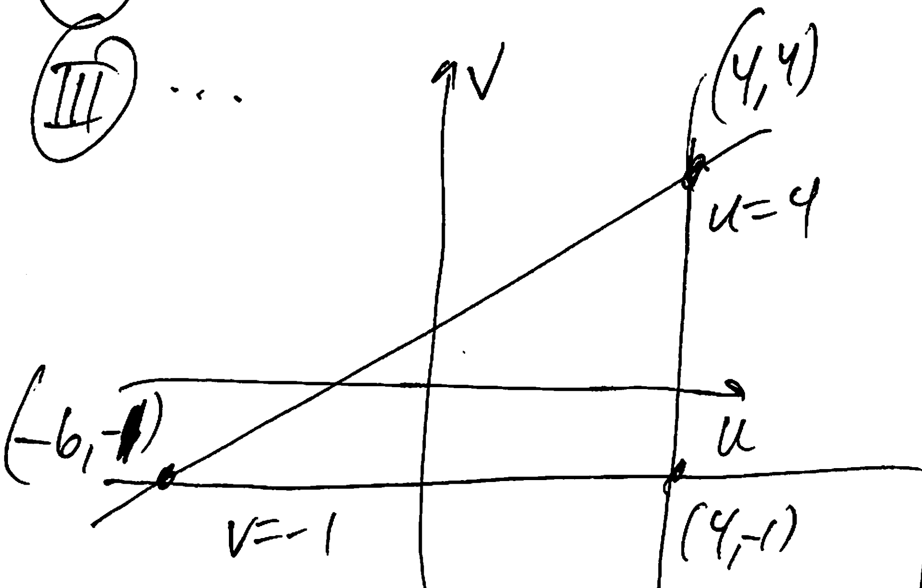
Plug in variable change to find new limits

I $\quad y = \cancel{0} - x + 4$

$$\tfrac{1}{2}(u-v) = -\tfrac{1}{2}(u+v) + 4$$
$$\Rightarrow u = 4$$

II $\quad v = -1 \quad$ (same procedure.)

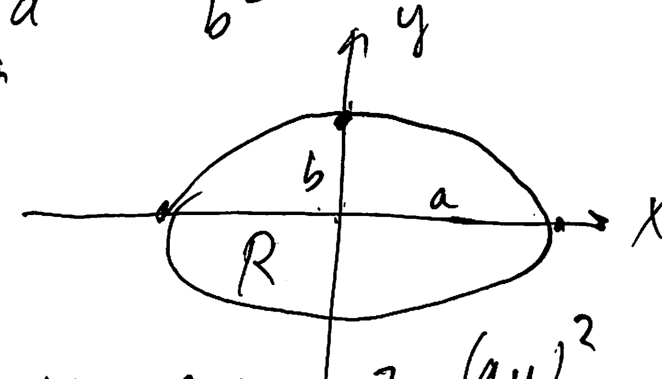
III $\quad \ldots$



now easy to finish!

application: area of ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \qquad a, b > 0$$



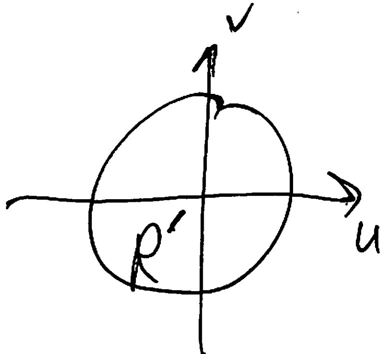
$$\left.\begin{aligned} x &= au \\ y &= bv \end{aligned}\right\} \quad \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = 1$$

$$\Updownarrow$$

$$u^2 + v^2 = 1$$

$\Longrightarrow$ in new coords, get unit circle u, v.

u, v:



Jacobian.

xy $\qquad\qquad\qquad\qquad$ uv

$$\iint_R dA = \iint_{R'} \boxed{}\, dA'$$

$dx\,dy$ $\qquad\qquad\qquad\qquad$ $du\,dv$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right|$$

$$= \left|\det\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right|$$

$$= |ab| = ab$$

$$\Rightarrow \iint_{R'} ab\,du\,dv$$

$$= ab \iint_{R'} du\,dv = \pi ab.$$

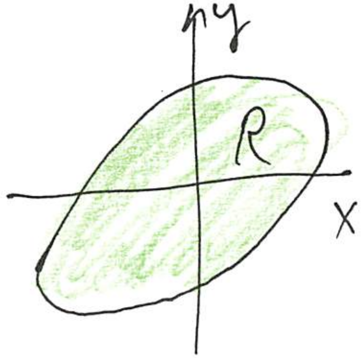
e.g. $\displaystyle\iint_R (x^2 - xy + y^2)\,dx\,dy.$

where $R = x^2 - xy + y^2 \le 2$

using
$$\left.\begin{aligned} x &= \sqrt{2}\,u - \sqrt{\tfrac{2}{3}}\,v \\ y &= \sqrt{2}\,u + \sqrt{\tfrac{2}{3}}\,v. \end{aligned}\right\} (*)$$

$x^2 - xy + y^2 = 2$

rotated ellipse.



Jacobian: $\left| \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \right|$

$= \left| \det \begin{pmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & +\sqrt{\frac{2}{3}} \end{pmatrix} \right|$

$= 4/\sqrt{3}.$

$dx\,dy = \frac{4}{\sqrt{3}}\,du\,dv.$

plug in (x) into integrand

$x^2 - xy + y^2 \Rightarrow 2u^2 + 2v^2.$

$\Rightarrow \iint_{R'} (2u^2 + 2v^2) \frac{4}{\sqrt{3}}\,du\,dv$

u,v coords.

R': $x^2 - xy + y^2 \le 2$

$\Updownarrow$

$2u^2 + 2v^2 \le 2$

$u^2 + v^2 \le 1.$

$\frac{8}{\sqrt{3}} \iint \underbrace{(u^2 + v^2)}_{r^2}\,du\,dv$

$u = r\cos\theta$

$v = r\sin\theta$

unit circle

$\Rightarrow \frac{8}{\sqrt{3}} \int_0^{2\pi} \int_0^1 r^2 \cdot r\,dr\,d\theta$

$= \boxed{\frac{4\pi}{\sqrt{3}}}.$