Ch 16 Vector calculus.

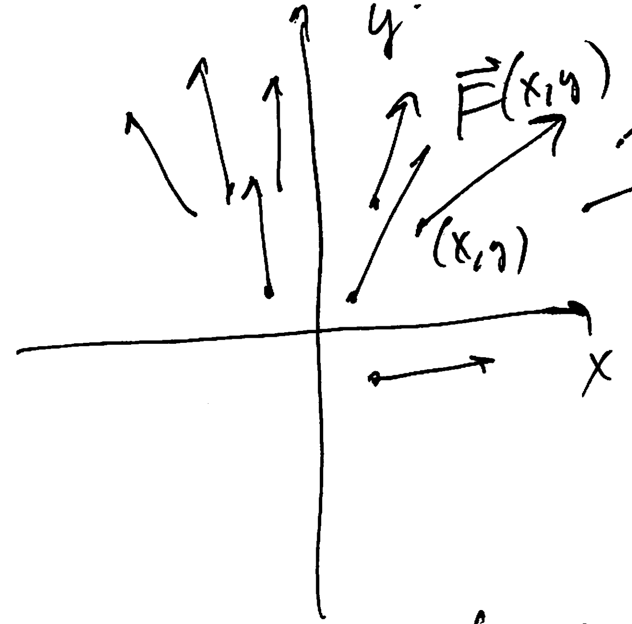
goal: higher-dimensional generalizations of the fundamental theorem of calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$F' = f.$$

16.1 Vector fields.

Def a vector field is a function that attaches a vector to each point in plane or in 3D.



Input : pt (x,y)
Output : vector $\vec{F}(x,y)$

graphing means drawing these vectors with tails at the input point.

These model force fields in applications e.g. forces due to wind at points in space.

e.g. gravitational forces in the presence of some body.

e.g. forces due to fluid flow.

e.g. $$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

i.e. have package of 2 functions of 2 variables

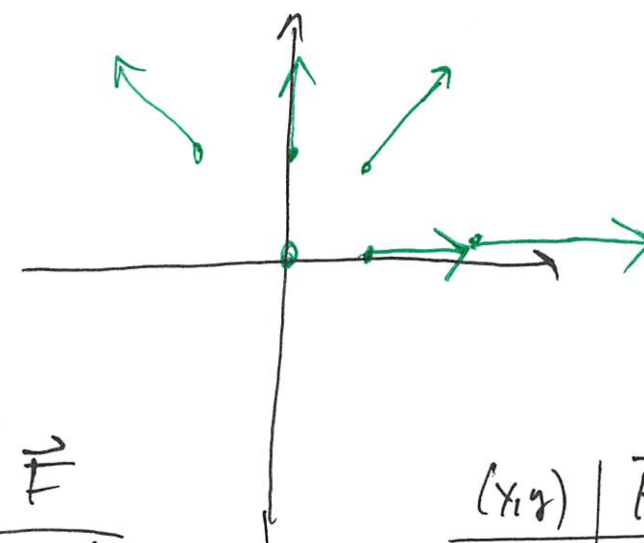
$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

$$= P(x,y,z)\,\hat{i} + Q(x,y,z)\,\hat{j} + R(x,y,z)\,\hat{k}$$

compare: vector valued fn VVF

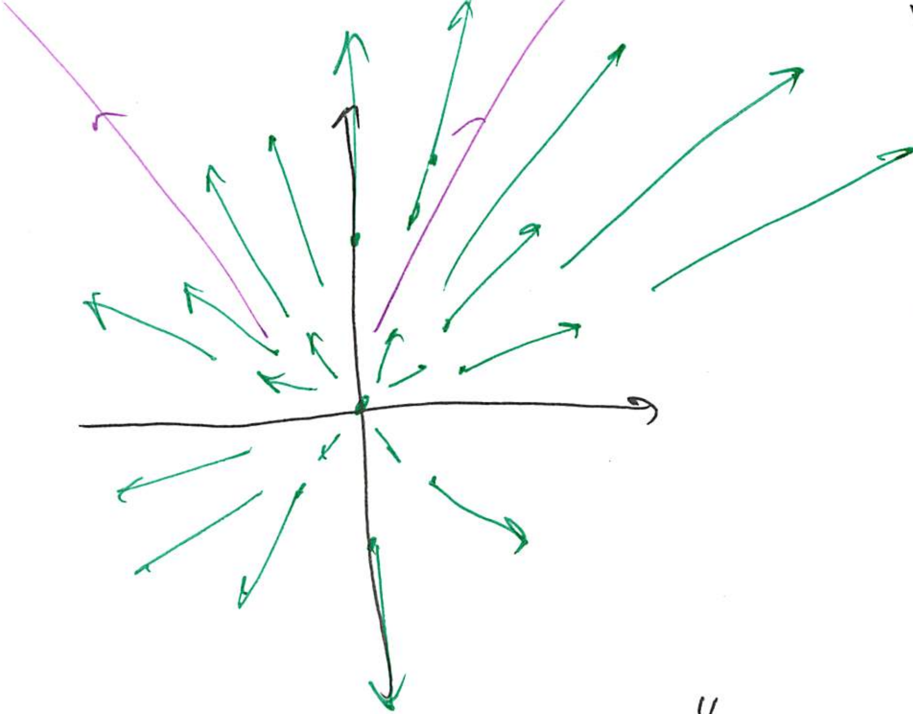
$$\vec{f}(t) = \langle x(t), y(t), z(t) \rangle$$

e.g. $\vec{F}(x,y) = \langle x, y \rangle$.



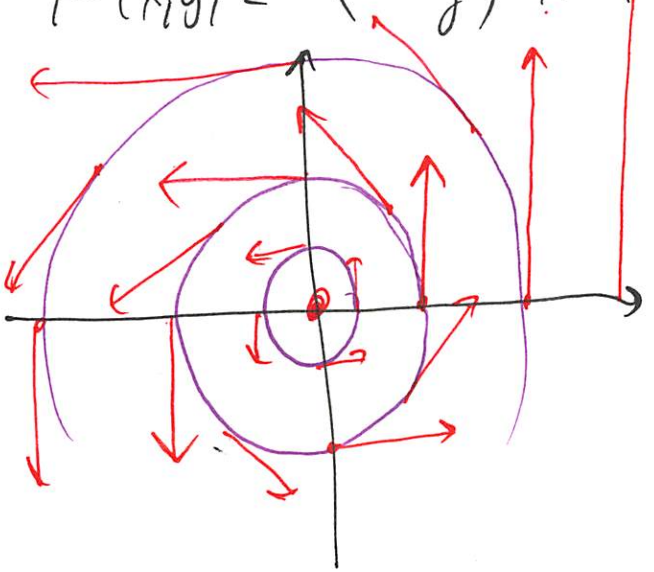
(x,y)	$\vec{F}$
$(0,0)$	$\vec{0}$
$(1,0)$	$\hat{i}$
$(2,0)$	$2\hat{i}$
$(0,1)$	$\hat{j}$

(x,y)	$\vec{F}$
$(1,1)$	$\hat{i}+\hat{j}$
$(-1,1)$	$-\hat{i}+\hat{j}$



"radial vector field"

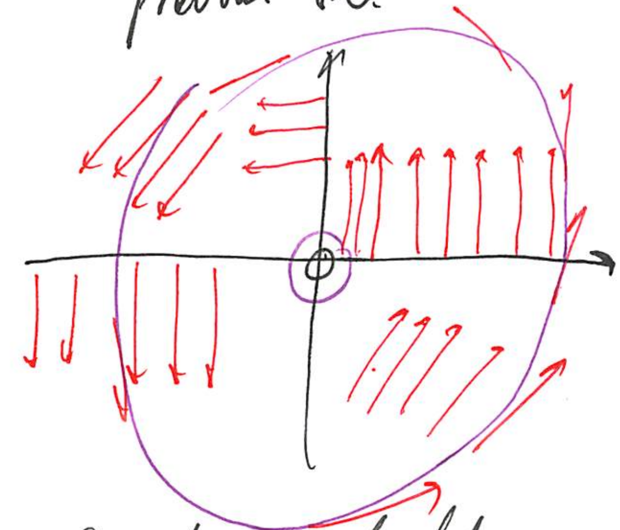
e.g. $\vec{F}(x,y) = \langle -y, x \rangle$



vectors tangent to same circle have the same length.

e.g. $\vec{F}(x,y) = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$

unit vector version of previous one.



$\vec{F}$.

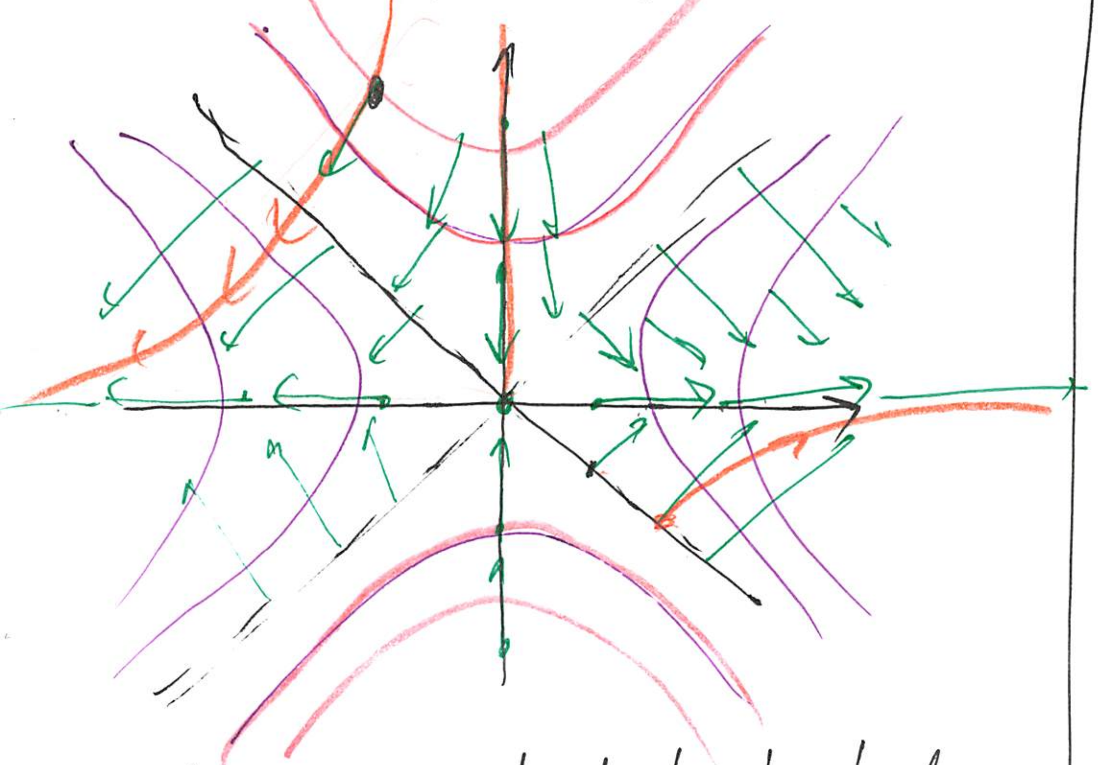
e.g. gradient field.

Start with function $f(x,y)$

Form gradient $\nabla f = \langle f_x, f_y \rangle$

e.g. $f = x^2 - y^2$

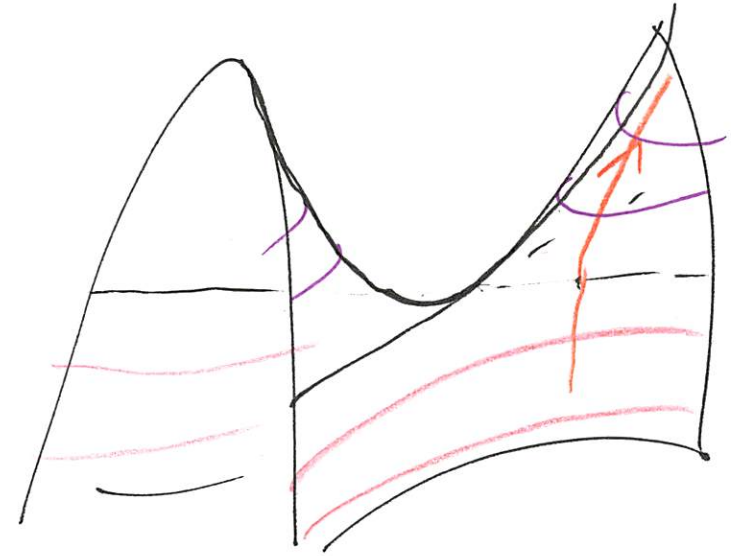
$\nabla f = \langle 2x, -2y \rangle$



Know: gradient $\perp$ to level curve of f.

$\rightarrow$ level curves are hyperplanes.

$z = x^2 - y^2$



Q: given a function f, get a V.F. using the gradient: $\bar{F} = \nabla f$.

Is every vector field the gradient of a function?

A: No.

e.g. $\langle x, y \rangle = \nabla f$

$f = \frac{1}{2}(x^2+y^2)$

e.g. $\langle -y, x \rangle$ is not ∇f for any f.

assume it is.

$\Longrightarrow f_x = -y$

$f_y = x$

must have: $f_{xy} = f_{yx}$

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(-y) = -1$

$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(x) = 1$ $\neq$

$\Rightarrow$ no such f.

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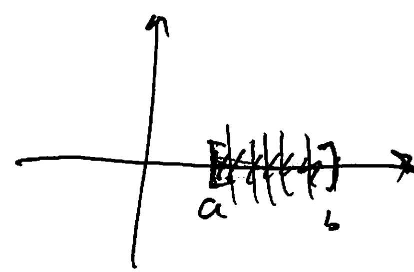
16.2 Line integrals.

There are integrals where the domain of integration is a curve in 2D or 3D.

i.e. domain is graph of a VVF.

~~eg~~ Calc II: $\int_a^b f(x)\,dx$

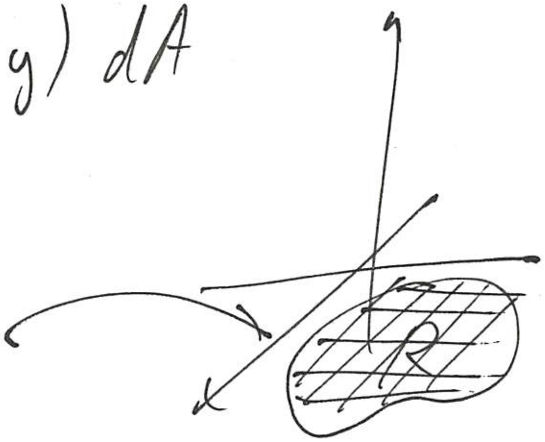
domain: $[a,b]$



flat.

double, triple ints similar.

$$\iint_R f(x,y)\, dA$$

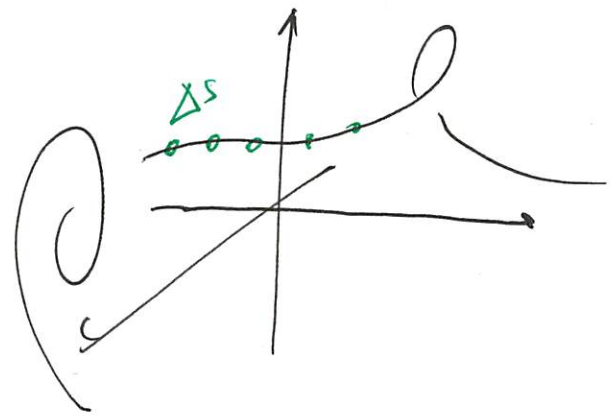


flat region

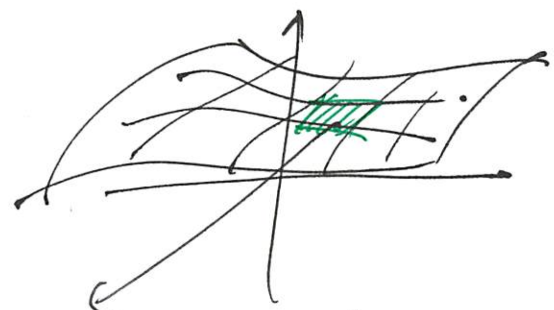
$dA = dx\,dy$

want more general integrals where the domains of integration are not flat.

e.g. arc length.



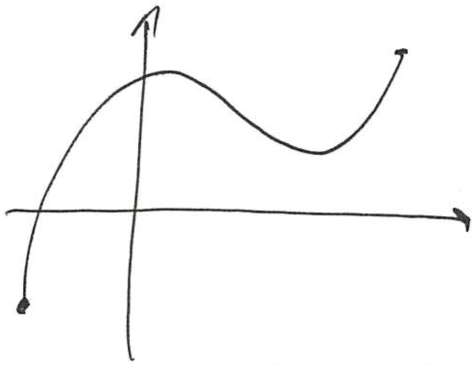
e.g. surface area.



1st examples: Line integrals.

Have curve C

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$
$$a \le t \le b$$



$f(x,y)$ function in the plane.

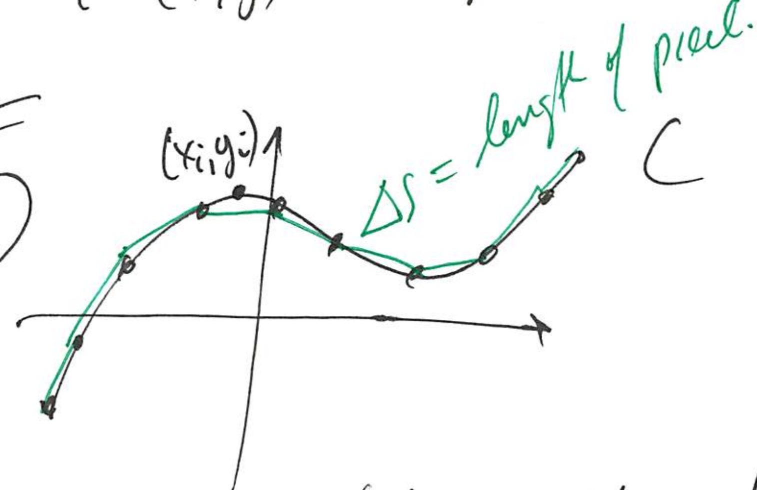
want to integrate this function along the curve C. "line integral"

In fact have 2 different types of line integrals.

(1) C curve
$f(x,y)$ function
"L.I. with respect to arc length"

(2) C curve.
$\vec{F}(x,y)$ vector field.
"Line integral"

(1)



$f(x,y)$ take sample points (x_i, y_i)

sum we want is

$$\sum_i f(x_i, y_i)\Delta s$$

take limit as $\Delta s \to 0$

get

$$\int_C f(x,y)\,ds$$

$C: \vec{r}(t) = \langle x(t), y(t)\rangle$

$a \le t \le b$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Same integrand we saw when we did arc length

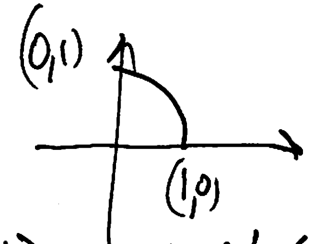
$$\int_C f(x,y)\,ds$$

$$= \int_a^b \underbrace{f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

normal 1 var integral variable is t.

eg $\displaystyle\int_C x\,ds$ where

$C = \frac{1}{4}$ circle



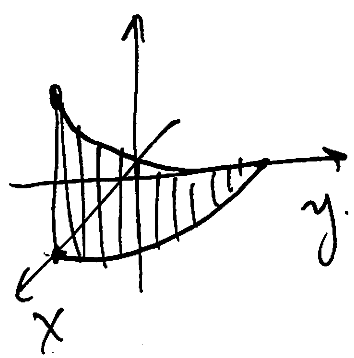
i.e. $\vec{r}(t) = \langle \cos t, \sin t\rangle$ $\quad 0 \le t \le \frac{\pi}{2}$.

$$ds = |\vec{r}\,'(t)|\,dt$$

$$= \sqrt{(-\sin)^2 + (\cos)^2}\,dt = 1\,dt.$$

$$\int_C x\,ds = \int_0^{\frac{\pi}{2}} \cos t\,dt$$

$$= \sin t \Big|_0^{\frac{\pi}{2}} = 1$$



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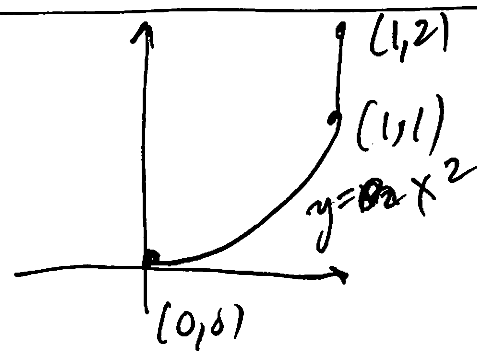
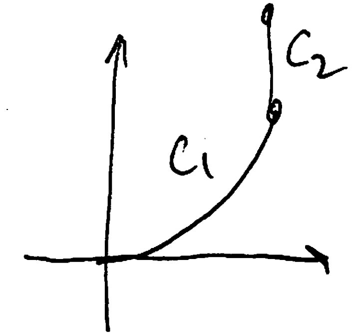
e.g. $C =$



$$\int_C 2x\,ds = ?$$

$$\boxed{\int_a^b f = \int_a^c f + \int_c^b f}$$



$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

C_1: $x = t$, $y = t^2$ $\quad 0 \le t \le 1$

C_2 = $x = 1$, $y = t$ $\quad 1 \le t \le 2.$

C_1: $ds = \sqrt{1 + 4t^2}\,dt$

$$\int_0^1 2t\sqrt{1+4t^2}\,dt$$

$u = 1 + 4t^2, \ldots$

C_2: $ds = \sqrt{1^2}\,dt = dt$

$$\int_0^1 2\,dt = 2.$$