curl / divergence

$\vec{F} = \langle P, Q, R \rangle$ 3D v.f.

$\operatorname{curl} \vec{F}$ new 3D v.f.

kind of derivative of $\vec{F}$

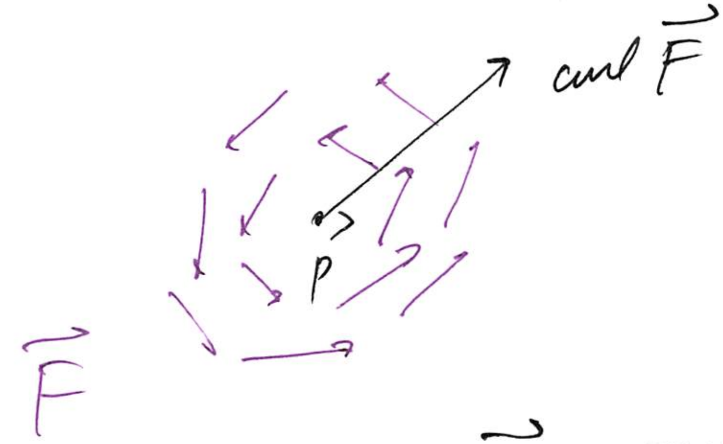
$\nabla = \langle \partial_x, \partial_y, \partial_z \rangle \quad \partial_x = \frac{\partial}{\partial x}$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

$$= \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix}$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$\operatorname{curl} \vec{F}$ new vector field that reflects the "circulation" in $\vec{F}$



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Remark: if $\vec{F} = \nabla \varphi$, i.e. is conservative, then $\operatorname{curl} \vec{F} = \vec{0}$

$\Rightarrow$ gives a test

$\operatorname{curl}(\nabla \varphi) = \vec{0}$

<u>Divergence</u>

another kind of derivative.

takes 3D vector fields to functions

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$\vec{F} = \langle P, Q, R \rangle$$

$$\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$$

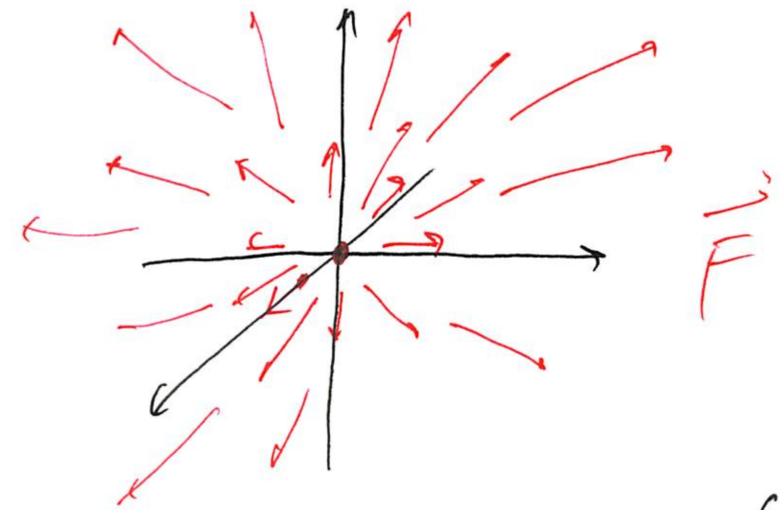
$$\operatorname{div} \vec{F} = \partial_x P + \partial_y Q + \partial_z R$$
$$= P_x + Q_y + R_z$$

e.g. $\vec{F} = \langle xy,\ z+y^2,\ 2xz \rangle$

$$\operatorname{div} \vec{F} = y + 2y + 2x$$
$$= 3y + 2x.$$

e.g. $\vec{F} = \langle \tfrac{1}{3}x,\ \tfrac{1}{3}y,\ \tfrac{1}{3}z \rangle$

$$\operatorname{div} \vec{F} = 1$$



Divergence looks like it's reflecting how much the vector field resemble the rays of a point source spreading out in space.

Have the big three grad, curl, div.

$\mathcal{F}$ = all functions in 3 variable

$\mathcal{X}$ = all vector fields in 3D

$$\mathcal{F} \xrightarrow{\text{grad.}} \mathcal{X} \xrightarrow{\text{curl}} \mathcal{X} \xrightarrow{\text{div}} \mathcal{F}$$

If we compose any 2 of these operations, in the order given, we automatically get 0.

$$\varphi \xmapsto{\text{grad}} \text{grad}\,\varphi \xmapsto{\text{curl}} \text{curl}(\text{grad}\,\varphi) = \vec{0}$$

i.e. $\text{div}(\text{curl}\,\vec{F}) = 0$

$$\vec{F} = \langle P, Q, R \rangle$$

$$\text{curl}\,\vec{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$\text{div}(\text{curl}\,\vec{F}) = \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = 0$$

Now we have our derivatives for vector calculus. need still another kind of integral to work with to ~~go~~ prove our generalizations of the fundamental theorem.

So far: usual 1D int

double integrals,
triple integrals.
line integrals.

Need: surface integrals.

— generalize double integrals by replacing region in the plane with regions on surfaces in 3D.

- generalize surface area integral we did.

Before we do this, we do

16.6 Parametric surfaces.

2D analogue of parametric curves, i.e. a 2D V.V.F.

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$a \le t \le b$

graph is a curve C in 3D.

For parametric surfaces, we need 2 independent parameters u, v.

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

Input: u, v output: x, y, z.

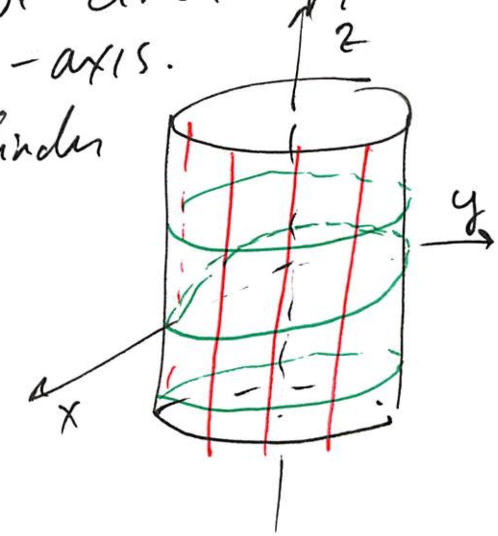
e.g. $\vec{r}(u,v) = \langle \overset{x}{\cos u}, \overset{y}{\sin u}, \overset{z}{v} \rangle$

$v = 0$ get unit circle in xy plane

Vary v : move circle up/down z-axis.

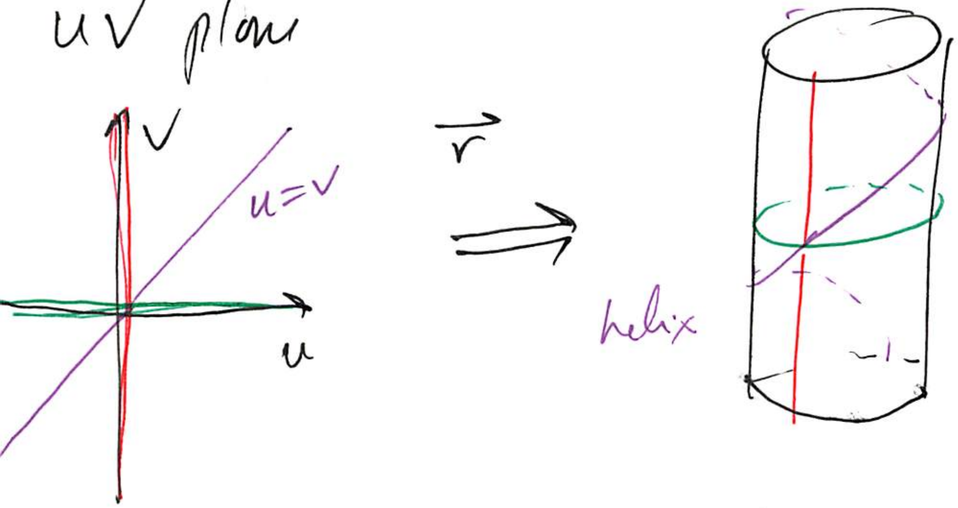
$\Longrightarrow$ get cylinder

graph of $\vec{r}(u,v)$



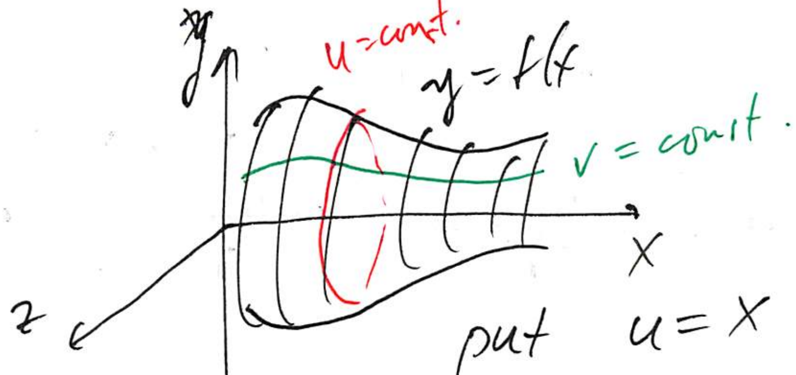
const v shapes.
const u shapes

uv plane



e.g. surface of revolution.

$y = f(x)$, rotate around the x-axis



put $u = x$

$$\vec{r}(u,v) = \langle u, f(u)\cos v, f(u)\sin v \rangle$$

e.g. helicoid

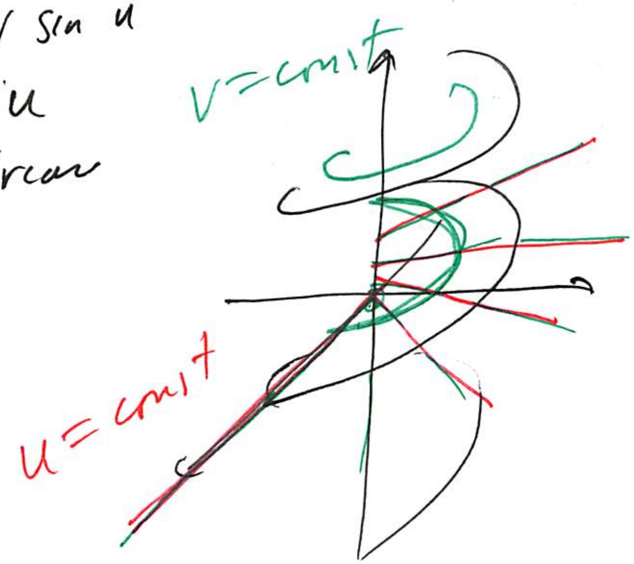
$$x = \cos u$$
$$y = \sin u$$
$$z = u$$

helix

take another parameter v and make

$$x = v\cos u \qquad v > 0$$
$$y = v\sin u$$
$$z = u$$

spiral staircase



For tangent planes, need normal vectors to the surface

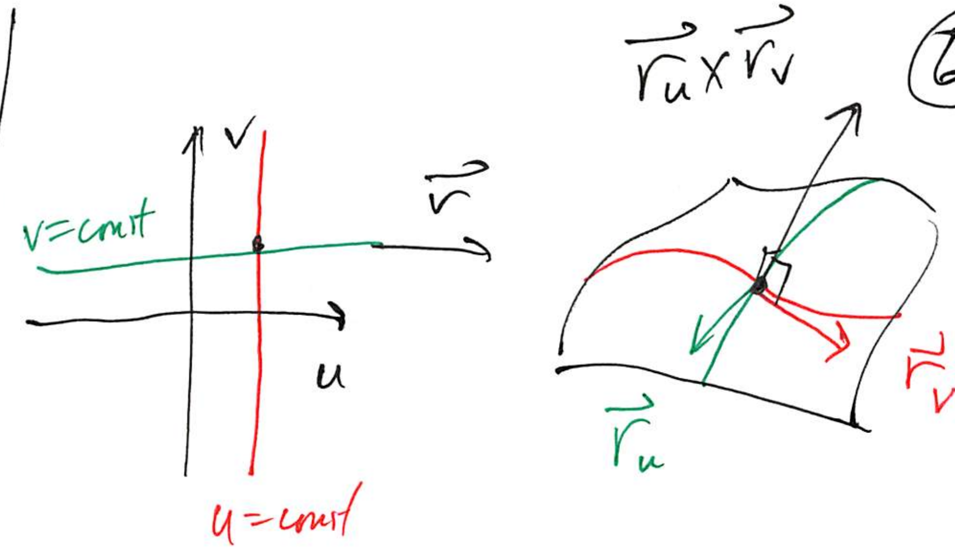
(u_0, v_0) want normal vector to $\vec{r}(u_0, v_0)$

$$\vec{r}_u := \langle x_u, y_u, z_u \rangle \Big|_{\substack{u=u_0 \\ v=v_0}}$$

gives a tangent vector to the graph.

$$\vec{r}_v := \langle x_v, y_v, z_v \rangle \Big|_{\substack{u=u_0 \\ v=v_0}}$$

get another



$\vec{r}_u \times \vec{r}_v$ gives a normal vector to tangent plane

eg suppose $S = \begin{array}{l} x = u^2 \\ y = v^2 \\ z = u + 2v. \end{array}$

find normal vector @ $(1,1,3)$
i.e. $u = v = 1$

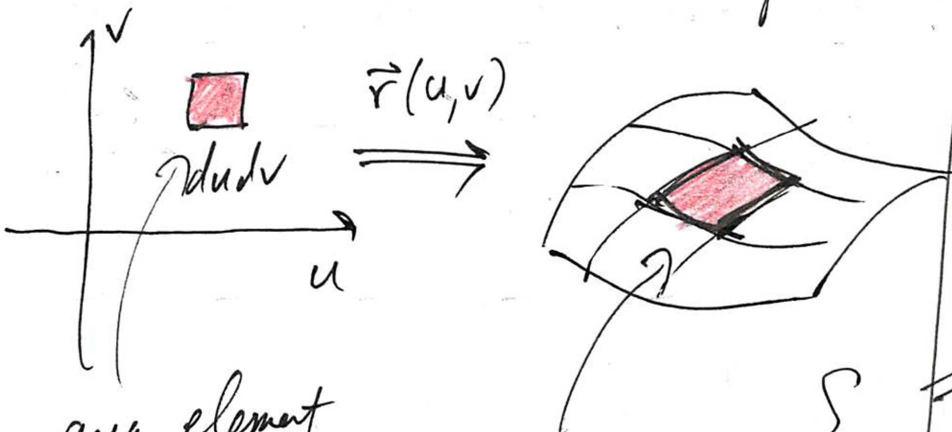
$$\left.\begin{array}{l} \vec{r}_u = \langle 2u, 0, 1 \rangle \\ \vec{r}_v = \langle 0, 2v, 2 \rangle \end{array}\right|_{u=1, v=1}$$

$\Longrightarrow \langle 2, 0, 1 \rangle, \langle 0, 2, 2 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle -2, -4, 4 \rangle$.

Now surface integrals

dS = element of surface area on surface S.



area element in uv plane

image of our tiny rectangle

dS = area of this image.

$$dS = \underbrace{\text{distortion factor}}\, du\,dv$$

ans: related to jacobian we discussed in Ch 15.

It turns out the answer is

$$dS = |\vec{r}_u \times \vec{r}_v|\, du\,dv.$$

Example: compute for a unit sphere.
need parametric eqns for unit sphere. get from spherical coordinates.

$x = \sin v \cos u \qquad 0 \le u \le 2\pi$

$y = \sin v \sin u \qquad 0 \le v \le \pi$

$z = \cos v$

(connection to spherical coords:
$\rho = 1, \ u = \theta, \ v = \varphi$)

goal: compute

$$|\vec{r}_u \times \vec{r}_v|$$