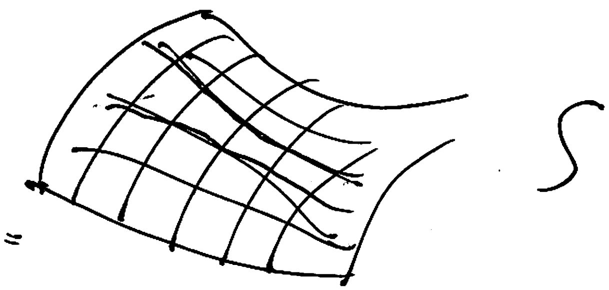
Last time:
parametric surface.

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$



dS = area element on surface

$$= \underbrace{\quad (*) \quad}\, du\,dv$$

distortion factor

$$= |\vec{r}_u \times \vec{r}_v|\, du\,dv$$

$$\vec{r}_u = \langle x_u, y_u, z_u \rangle$$
$$\vec{r}_v = \langle x_v, y_v, z_v \rangle$$

e.g. area of S

$$= \iint_S dS = \iint_{(u,v)} |\vec{r}_u \times \vec{r}_v|\, du\,dv.$$

e.g. S is the unit sphere

$$x = \sin v \cos u \qquad 0 \le v \le \pi$$
$$y = \sin v \sin u \qquad 0 \le u \le 2\pi$$
$$z = \cos v \qquad \begin{pmatrix} \rho = 1, \\ u = \theta, \\ v = \varphi \end{pmatrix}$$

compute $|\vec{r}_u \times \vec{r}_v|$

$$\vec{r}_u = \langle \sin v(-\sin u), \sin v \cos u, 0 \rangle$$
$$\vec{r}_v = \langle \cos v \cos u, \cos v \sin u, -\sin v \rangle$$

$\vec{r}_u \times \vec{r}_v =$

$$\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \text{—} & \vec{r}_u & \text{—} \\ \text{—} & \vec{r}_v & \text{—} \end{pmatrix}$$

$= \langle -\sin^2 v \cos u, \; -\sin^2 v \sin u, \; -\cos v \sin v \rangle$

$|\vec{r}_u \times \vec{r}_v| =$

$= \sqrt{(\ )^2 + (\ )^2 + (\ )^2}$

$=$ use trig Ids

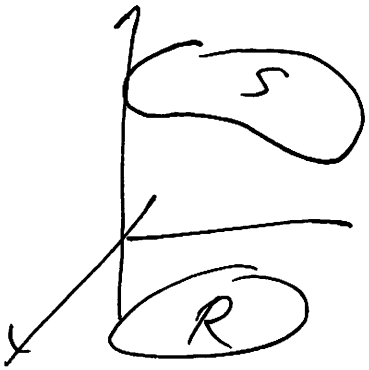
$= \sin v$.

$dS = \sin v \, du \, dv$.

(recall: spherical coords
$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$)

$$\iint_S dS = \int_0^{\pi}\int_0^{2\pi} \sin v \, du \, dv = 4\pi.$$

e.g. special case: S is the graph of a function $z = g(x,y)$.



$z = g(x,y)$.

From before: surface area was given by

$$\iint_R \sqrt{1+g_x^2+g_y^2}\,dx\,dy.$$

i.e. distortion factor was

$$\sqrt{1+g_x^2+g_y^2}$$

We can also see this using $|\vec{r}_u \times \vec{r}_v|$.

To get parametric eqns, take

$$\vec{r}(u,v) = \langle u, v, g(u,v) \rangle.$$

$$\vec{r}_u = \langle 1, 0, g_u \rangle$$

$$\vec{r}_v = \langle 0, 1, g_v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & g_u \\ 0 & 1 & g_v \end{pmatrix}$$

$$= \langle -g_u, -g_v, 1 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{1+g_u^2+g_v^2}$$

same as before.

16.7 Surface integrals

Recall that we have 2 kinds of line integral.

C curve

① $\int_C f(x,y,z)\, ds$ line integral with respect to arc length.

f function of 3 variables

② $\int_C \vec{F} \cdot d\vec{r}$ line integral.

$\vec{F}$ = vector field.

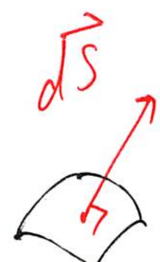
For surface integrals, also have 2 different kinds that parallel what we did for line integral. S = surface.

① $\iint_S f(x,y,z)\, dS$ surface integral

f function

$dS = |\vec{r}_u \times \vec{r}_v|\, du\, dv$

② $\iint_S \vec{F} \cdot d\vec{S}$ flux integral.



$\vec{F}$ vector field.

$d\vec{S}$ small vector $\perp$ to surface.

For ①, proceed as we did for parametric surfaces. just incorporate the function f into the integrand.

$$dS = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

where $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is a parametric representation of our surface.

$$\iint_S f(x,y,z)\, dS = \iint_{(u,v)} f(x(u,v), y(u,v), z(u,v))\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

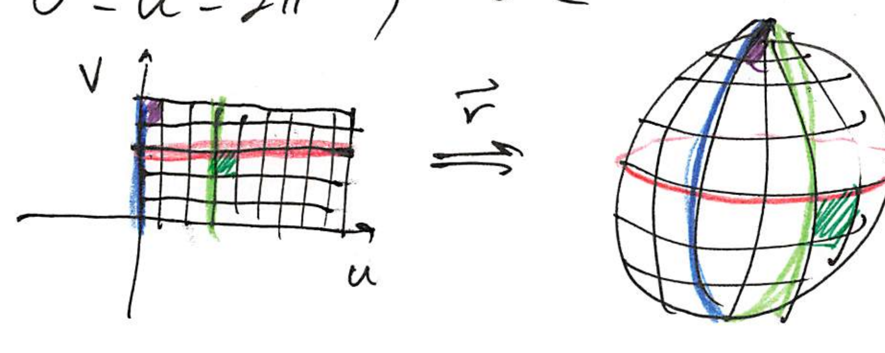
RHS is now an ordinary double integral in the variables u, v.

e.g. S = unit sphere.

$$\iint_S x^2\, dS \qquad (*)$$

$$\vec{r}(u,v) = \begin{cases} x = \sin v \cos u \\ y = \sin v \sin u \\ z = \cos v \end{cases}$$

$0 \le u \le 2\pi, \quad 0 \le v \le \pi$



recall $|\vec{r}_u \times \vec{r}_v| = \sin v$.

$\Rightarrow dS = \sin v \, du \, dv$

(*) $\iint_S x^2 \, dS = \iint_S (\sin v \cos u)^2 dS$

$= \int_0^{\pi}\int_0^{2\pi} \sin^3 v \cos^2 u \, du \, dv$

$= \left(\int_0^{\pi} \sin^3 v \, dv\right)\left(\int_0^{2\pi} \cos^2 u \, du\right)$

$= \cdots = 4\pi/3$.

e.g. $\iint_S y \, dS$

S: $z = x + y^2 =: g(x,y)$.
$0 \le x \le 1$
$0 \le y \le 2$.

$u = x$, $v = y$, $z = u + v^2$

$\vec{r} = \langle u, v, u+v^2 \rangle$.

$|\vec{r}_u \times \vec{r}_v| = \sqrt{1 + g_u^2 + g_v^2}$

$= \sqrt{1^2 + 1^2 + 4v^2} = \sqrt{2 + 4v^2}$

$\iint_S y \, dS = \int_0^2 \int_0^1 v\sqrt{2+4v^2} \, du \, dv$

$= \cdots = 13\sqrt{2}/3$

Flux integral

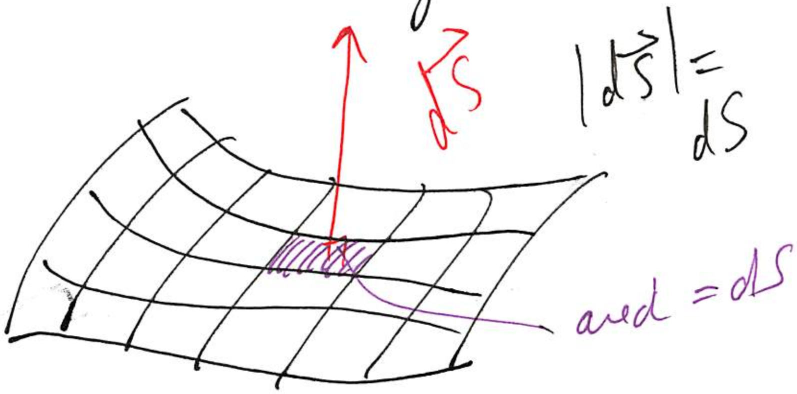
$$\iint_S \vec{F} \cdot d\vec{S} \qquad (*)$$

$\vec{F}$ = vector field in 3D

$= \langle P, Q, R \rangle$

S parametric surface

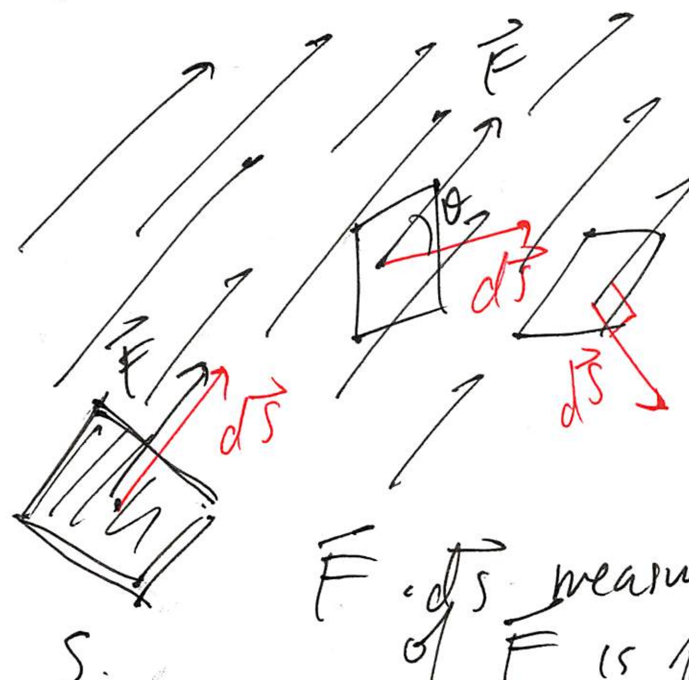
$d\vec{S}$ = vector $\perp$ to S with length dS



We will be able to convert everything to a usual integral in u, v.

What is $(*)$ computing?

Imagine $\vec{F}$ is a constant vector field and S is a rectangle

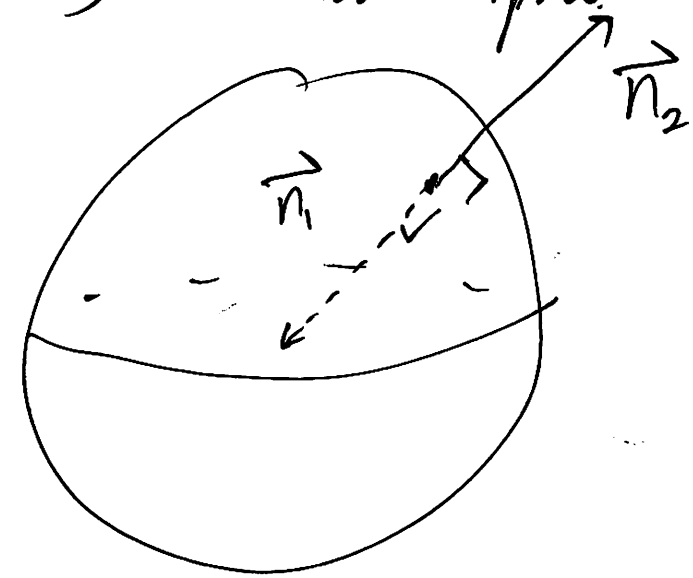


$\vec{F} \cdot d\vec{S}$ measures how much of $\vec{F}$ is flowing through S.

$\iint_S \vec{F} \cdot d\vec{S}$ computes something about the net flow of $\vec{F}$ through the surface S.

To compute the integral, we must choose a direction for $d\vec{S}$. It's a normal vector to the surface and we must be able to choose it consistently over the surface. The surface must be what's called <u>orientable</u>

e.g. S = unit sphere



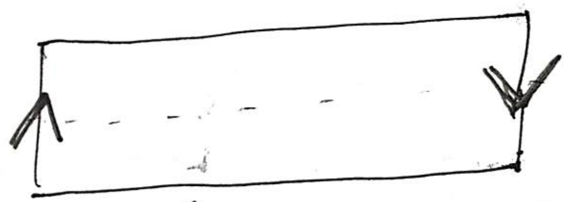
have 2 possibilities at any point on S for a normal vector.

1. inward pointing $\vec{n}_1$
2. outward pointing $\vec{n}_2$

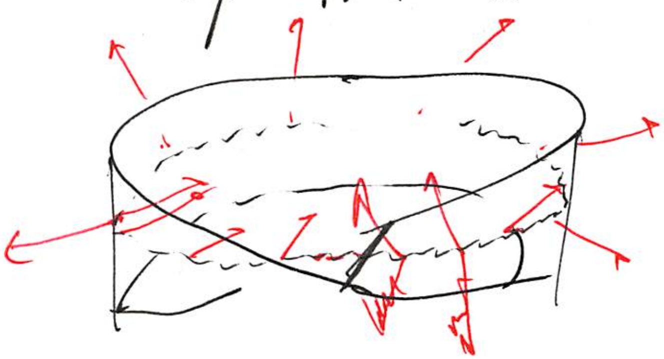
We must pick one over all of S. There are surfaces where one cannot consistently do this.

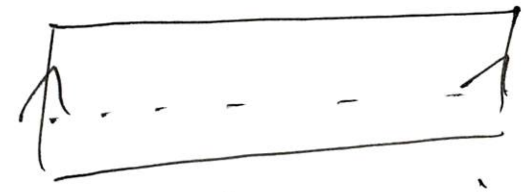
Such surfaces are called non orientable.

eg Möbius strip.



glue strip into a band by matching up the arrow.





more discussion in text.

We can't allow such surfaces in ~~of~~ our flux integrals.

The $d\vec{S}$ can be taken to be $\pm(\vec{r}_u \times \vec{r}_v)\,du\,dv$.

This vector has right length.

$|\vec{r}_u \times \vec{r}_v|\,du\,dv = dS$

choose sign depending on the problem.