Flux integrals

$\vec{F}$ vector field.

S $\vec{r}(u,v)$ parametric surface.

$$\iint_S \vec{F} \cdot d\vec{S}$$

$d\vec{S}$ $\perp$ surface
has length dS
(area element on S).

$$\pm d\vec{S} = \vec{r}_u \times \vec{r}_v \, du\,dv$$

$\vec{r}_u = \langle x_u, y_u, z_u \rangle$

$\vec{r}_v = \langle x_v, y_v, z_v \rangle$

$x = x(u,v)$
$y = y(u,v)$
$z = z(u,v)$
(fns appearing in $\vec{r}(u,v)$).

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{(u,v)} \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\,dv$$

e.g. Find flux of

$\vec{F} = \langle z, y, x \rangle$

across the unit sphere using outward pointing normal.
(want $d\vec{S}$ to be pointing outward)

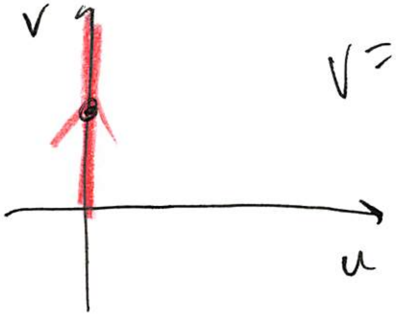
recall for unit sphere

$x = \sin v \cos u$ $\quad 0 \le v \le \pi$
$y = \sin v \sin u$ $\quad 0 \le u \le 2\pi$
$z = \cos v$

$$\vec{r}_u \times \vec{r}_v = \langle -\sin^2 v \cos u, -\sin^2 v \sin u, -\sin v \cos v \rangle$$

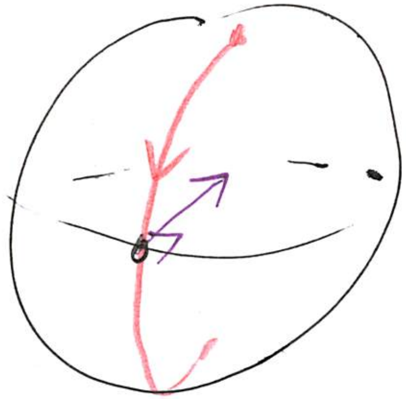
this is $\pm d\vec{S}$.

Q: inward or outward?



$v=\frac{\pi}{2},\ u=0$

$\vec{r}$



plug in, get

$\vec{r}_u \times \vec{r}_v = \langle -1, 0, 0 \rangle$

This is inward, so we want

$-\vec{r}_u \times \vec{r}_v = \vec{r}_v \times \vec{r}_u =$

$\langle \sin^2 v \cos u,\ \sin^2 v \sin u,\ \sin v \cos v \rangle$

$\nearrow d\vec{S}$

$$\iint_S \vec{F} \cdot d\vec{S}$$

$\vec{F} = \langle z, y, x \rangle$

on S,

$x = \sin v \cos u$

$y = \sin v \sin u$

$z = \cos v$

$$= \iint_{(u,v)} \vec{F} \cdot (\vec{r}_v \times \vec{r}_u)\, du\, dv$$

$$= \int_0^{2\pi} \int_0^{\pi} \cancel{\sin^2 \cos^2 u} \left(\sin^2 v \cos u \cos v + \sin^3 v \sin^2 u + \sin^2 v \cos u \cos v \right) \cancel{du\, dv}\ dv\, du$$

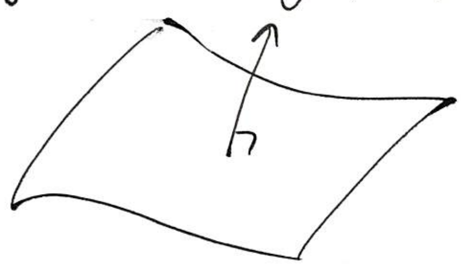
$$= \boxed{\frac{4\pi}{3}}$$

e.g. If S is the graph of $z = g(x,y)$

we can take x, y to be the ~~parameter~~ parameters, and if $\vec{F} = \langle P, Q, R \rangle$, then our final integral is

$$\pm \iint \left(-Pg_x - Qg_y + R \right) dx\,dy.$$

$$\vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1 \rangle$$



points up because z component is 1

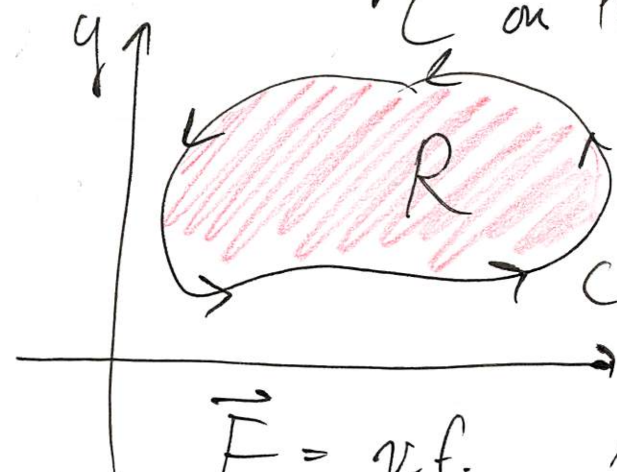
Stokes's Thm

Generalization of Green's Theorem to parametric surfaces in 3D.

Recall Green's Thm.

C = simple closed curve

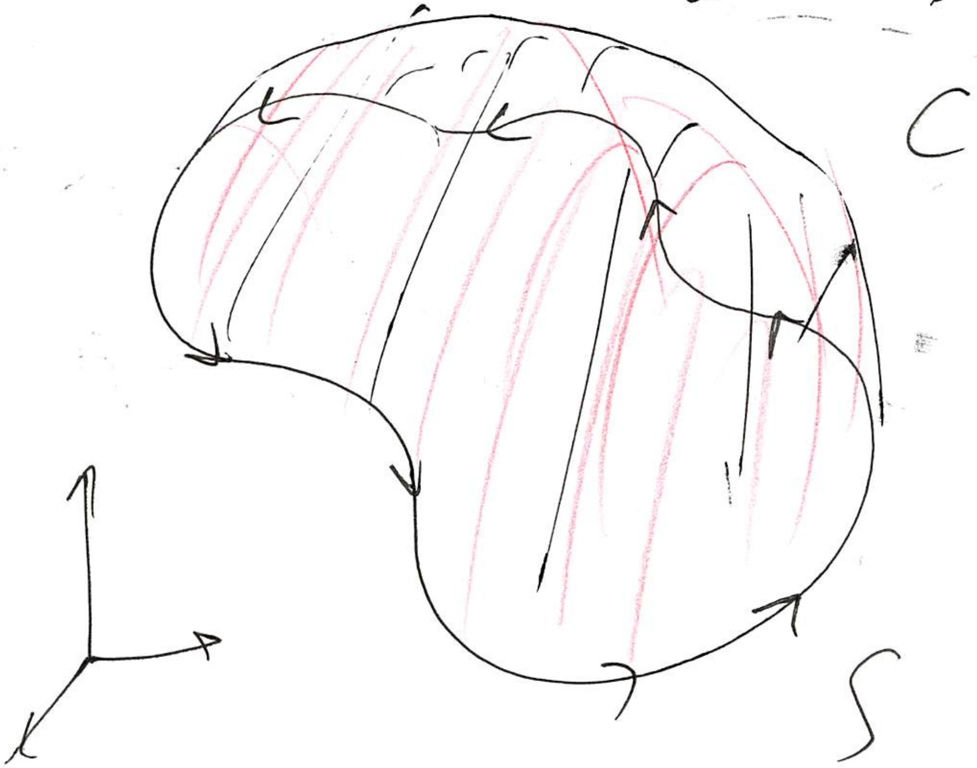
R = region bounded by C on the left side.



$\vec{F}$ = v.f. $= \langle P, Q \rangle$

G.T.: $\oint_C \vec{F}\cdot d\vec{r} = \iint_R (Q_x - P_y)\, dA$

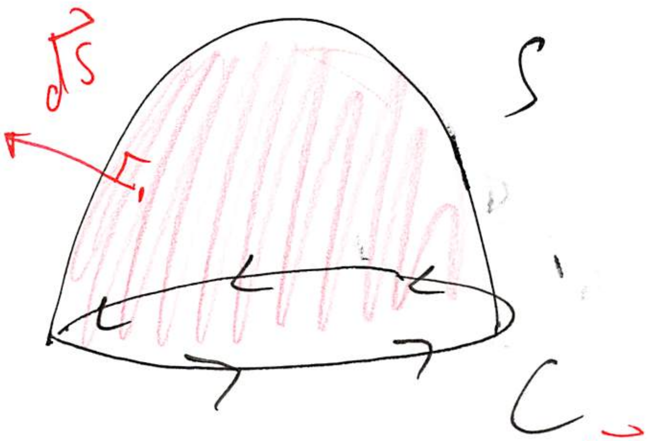
Stokes's Thm: make this 3D. C simple closed curve in 3D

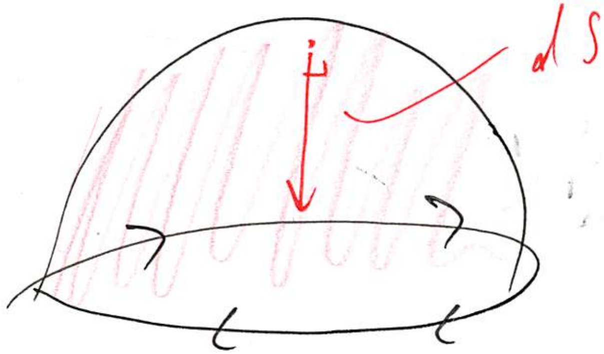


S = <u>any</u> orientable surface with boundary curve C.

assume C and S are compatibly oriented. This means the direction of travel on C is determined by $\vec{dS}$ and the right hand rule:

Thumb	$\vec{dS}$
Fingers	C





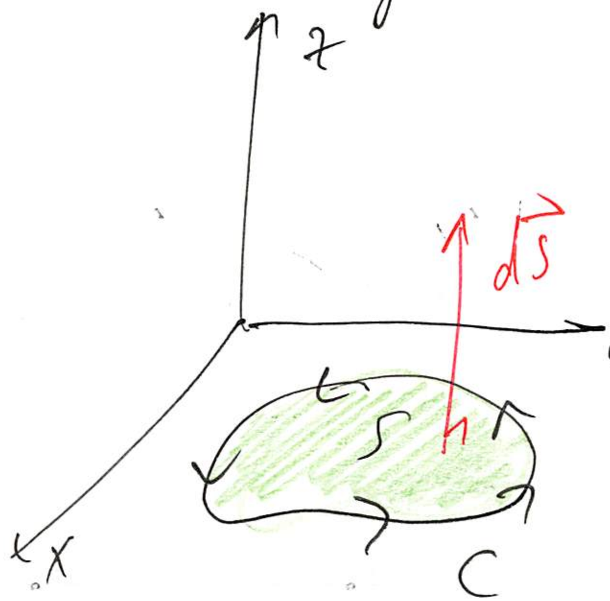
Stokes's Thm

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\operatorname{curl}\vec{F})\cdot d\vec{S}$$

Green's theorem: (5)

$\vec{F} = \langle P, Q, 0 \rangle$. P, Q only depend on x, y.

S = region in xy plane.



$$d\vec{S} = \hat{k}\, dx\, dy.$$

$$\operatorname{curl}\vec{F} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{pmatrix}$$

$$= \langle 0, 0, Q_x - P_y \rangle.$$

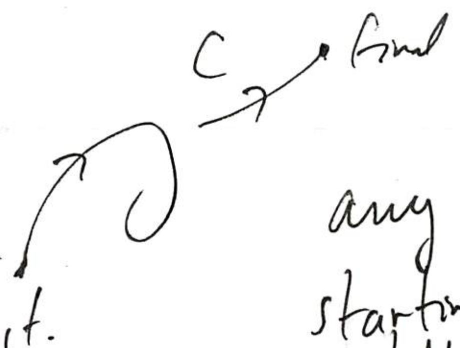
$$\operatorname{curl}\vec{F}\cdot d\vec{S} = (Q_x - P_y)\,dx\,dy$$

Like an analogue of fundamental theorem for line integrals.

φ = potential function

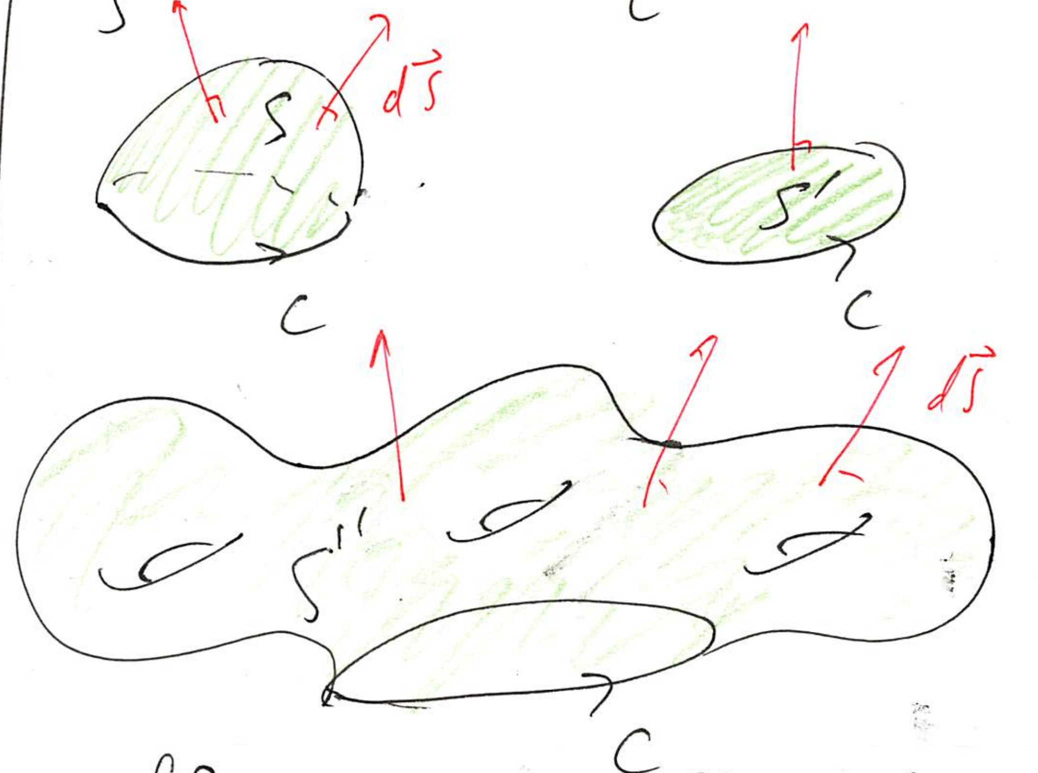
$\nabla\varphi$ = vector field.

$$\int_C \nabla\varphi\cdot d\vec{r} = \varphi(\text{final}) - \varphi(\cancel{\text{begin}}\ \text{init.})$$



any C starting and stopping at these points works

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$



$$\iint_S \operatorname{curl}F\cdot d\vec{S} = \iint_{S'} \operatorname{curl}F\cdot d\vec{S}$$

$$= \iint_{S''} \operatorname{curl}F\cdot d\vec{S}$$

e.g. $\vec{F} = \langle -y^2, x, z^2 \rangle$

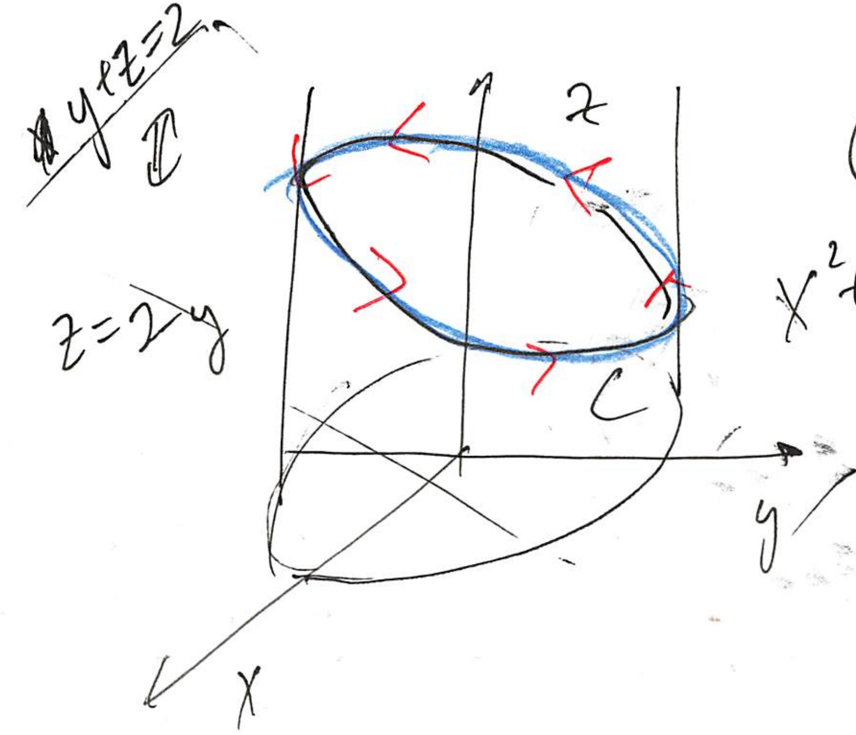
C = intersection of $y+z=2$

$x^2+y^2=1$

orient C to make the $+$ z-axis compatible.

$\oint_C \vec{F} \cdot d\vec{r} = ?$

Stokes's Thm. Need a surface S with boundary C.



parametric eqns for C are

$\vec{r}(t) = \langle \cos t, \sin t, 2-\sin t \rangle$

$0 \le t \le 2\pi$

$d\vec{r} = \langle -\sin t, \cos t, -\cos t \rangle dt$

$\vec{F} = \langle -\sin^2 t, \cos t, (2-\sin t)^2 \rangle$

$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \sin^3 t + \cos^2 t - \cos t (2-\sin t)^2 \, dt$

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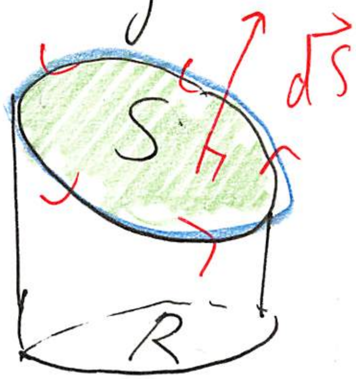
use the plane to get surface S with bounding curve C



$y+z=2$

$\Rightarrow z=2-y$

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = ?$

S is a graph: $g(x,y) = 2-y$.

$\perp$ vector given by

$\langle -g_x, -g_y, 1\rangle$

$= \langle 0, 1, 1\rangle$

This one is pointing up.

$\Rightarrow d\vec{S} = \langle 0,1,1\rangle\, dx\,dy.$

curl $\vec{F}$

$$= \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y^2 & x & z^2 \end{pmatrix}$$

$= \langle 0, 0, 1+2y\rangle.$

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_R (1+2y)\,dx\,dy$$

use polar $\displaystyle = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\, r\,dr\,d\theta$

$= \cdots = \boxed{\pi}.$