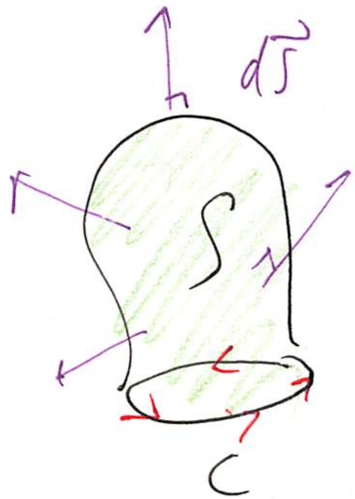
Stokes's Thm

S oriented surface

C boundary curve of S, directed compatibly with the orientation. (RH rule)

$\vec{F}$ vector field.

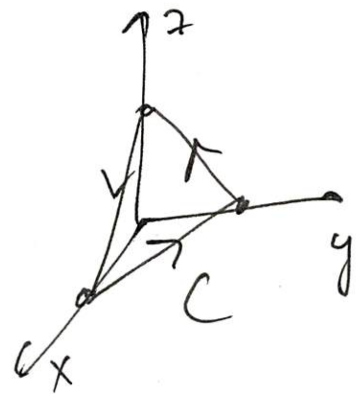


$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}}$$

e.g. $\oint_C \vec{F}\cdot d\vec{r}$

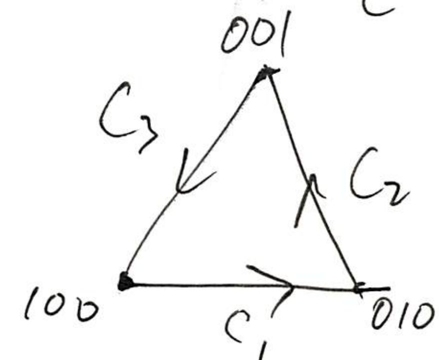
$\vec{F} = \langle z^2, y^2, x \rangle$.

C goes around the triangle with verts $(1,0,0), (0,1,0), (0,0,1)$ in this order.



could compute directly:

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$



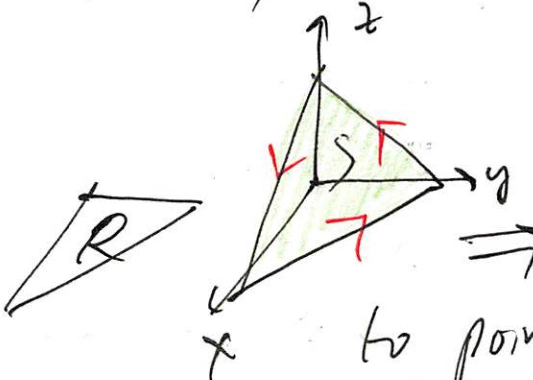
use S.T. instead.

$$\operatorname{curl}\vec{F} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z^2 & y^2 & x \end{pmatrix}$$

$$= \langle 0, 2z-1, 0 \rangle.$$

$$= (2z-1)\,\hat{j}$$

For S, can take triangle



plane is $x+y+z=1$

$\Rightarrow$ we need $d\vec{S}$ to point away from the origin.

need parametric eqn for S. write as a graph over x,y plane.

$z = 1-x-y$. (take x,y for params)

$d\vec{S} = \vec{r}_x \times \vec{r}_y\, dx\,dy$ up to sign.

$\vec{r}(x,y) = \langle x, y, 1-x-y \rangle$.

recall that if $z = g(x,y)$, then $\vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1 \rangle$

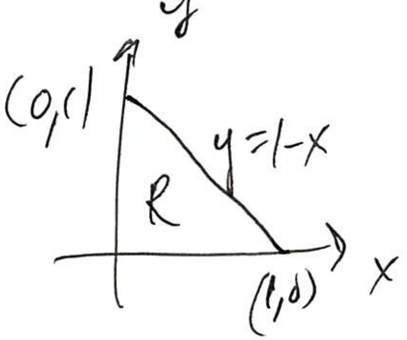
$$= \langle 1, 1, 1 \rangle$$

pointing away from origin ✓

$$\Longrightarrow \quad d\vec{S} = \langle 1,1,1 \rangle\, dx\,dy.$$

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \iint_R (2z-1)\, \cancel{dx\,dy}\, dA$$

$$= \iint_R \big(2(1-x-y)-1\big)\, dA$$

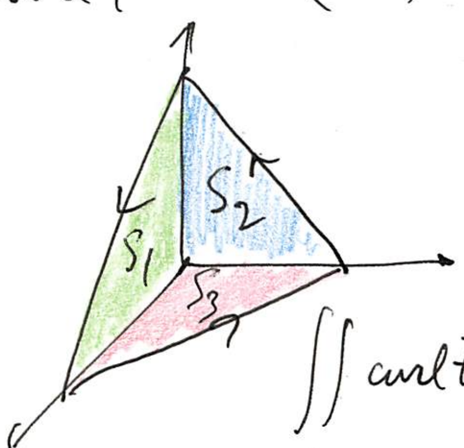


$0 \le x \le 1$
$0 \le y \le 1-x$

$$\Rightarrow \int_0^1 \int_0^{1-x} (1-2x-2y)\,dy\,dx = \boxed{-\frac{1}{6}}$$

another choice for S:

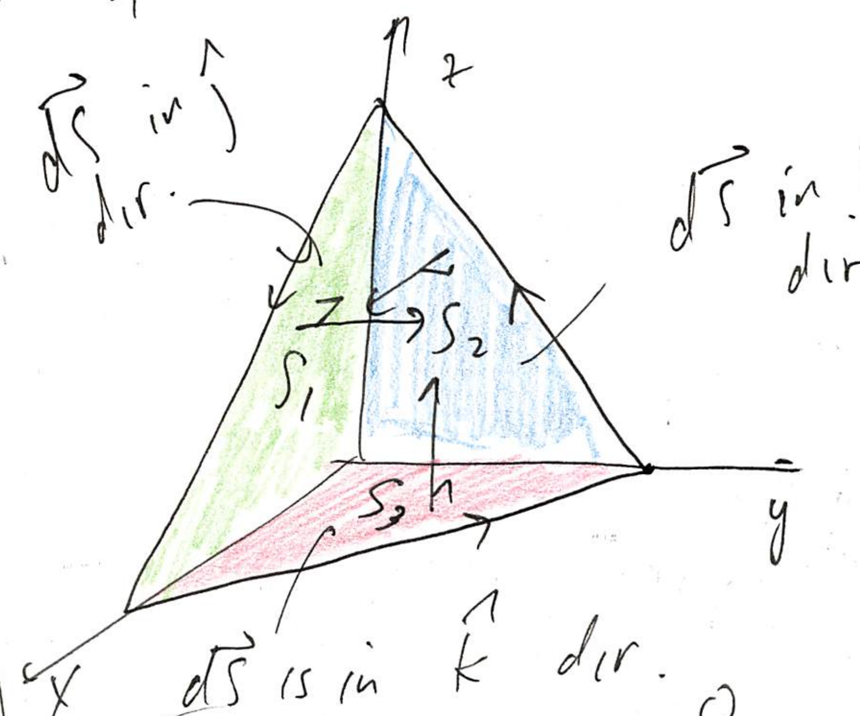
$\text{curl}\,\vec{F} = \langle 0, 2z-1, 0 \rangle$



$S = S_1 \cup S_2 \cup S_3$

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} =$$

$$= \iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S} + \iint_{S_2} \cdots + \iint_{S_3} \cdots$$



$$\iint_{S_1} \cdots + \cancelto{0}{\iint_{S_2} \cdots} + \cancelto{0}{\iint_{S_3} \cdots}$$

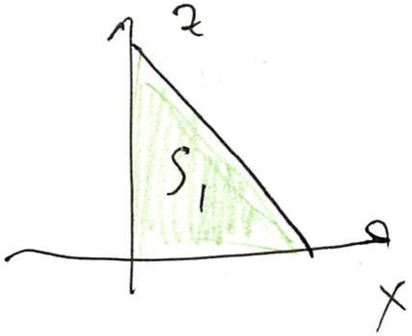
only possible contribution

$\Rightarrow$ get slightly better double integral!

$$d\vec{S} = dx\,dz\,\hat{j}$$

and $\vec{F}\cdot d\vec{S} = (2z-1)\,dx\,dz$

region in the plane is just S_1.



$0 \le x \le 1$
$0 \le z \le 1-x$

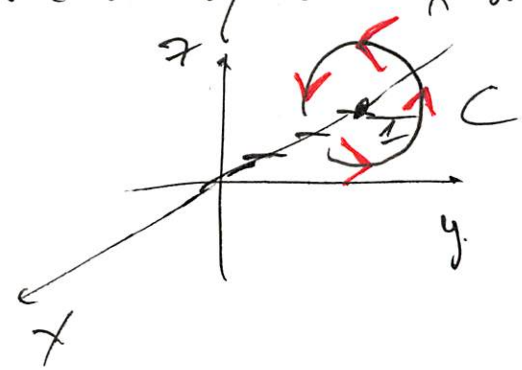
$$\int_0^1 \int_0^{1-x} (2z-1)\,dz\,dx$$

$$= -\frac{1}{6}$$

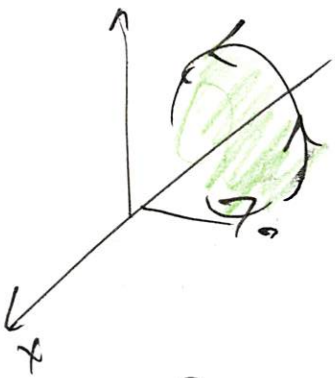
e.g. $\oint_C \vec{F}\cdot d\vec{r}$

$$\vec{F} = \langle x^2, -4z, xy \rangle.$$

C circle of radius 1.
$\perp$ to x axis
Center at $x=-3$ on x-axis.
Oriented counterclockwise when looking in direction of $-$ x-axis.



use Stokes. S = solid disk that fills in C.



$d\vec{S}$ must point in $\hat{i}$ + x direction

$$\text{curl}\,\vec{F} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & -4z & xy \end{pmatrix}$$

$$= \langle x+4, -y, 0 \rangle$$

$$\text{curl}\,\vec{F}\cdot d\vec{S} = (x+4)\,dS$$

where dS = area elt of surface.

on S, $x = -3$

$$\Rightarrow \text{curl}\,\vec{F}\cdot d\vec{S} = dS$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S dS = \text{area}(S) = \boxed{\pi}$$

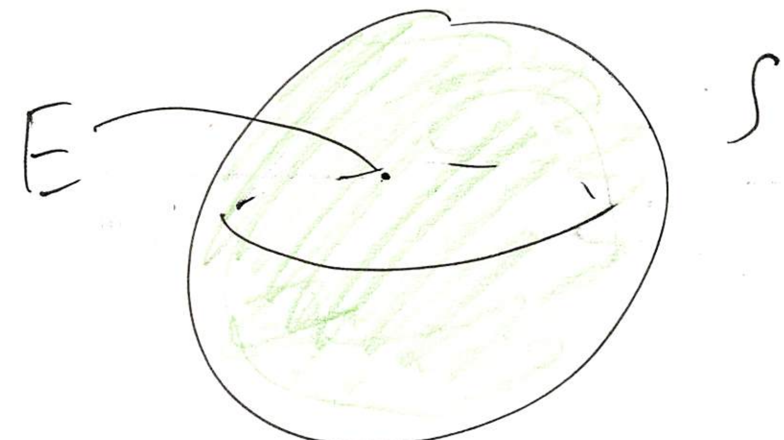
Divergence Theorem
a.k.a Gauss's Thm

relates a surface integral on a closed orientable surface to a triple integral on the interior E of the surface.

S orientable $\Leftrightarrow$ S has an inside and an outside
$\Leftrightarrow$ can consistently choose a $\perp$ vector to S.

S closed $\Leftrightarrow$ S has no boundary curve.

e.g. S = unit sphere in 3D.
S is closed and orientable.

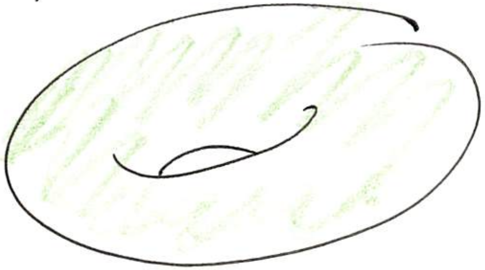


no boundary curve.

eg S = spherical cap

orientable
not closed.

e.g. torus



closed, orientable

orientable, not closed.

e.g. there do exist surfaces that are closed and not orientable.
They cannot be realized in 3D.
e.g. Klein bottle.
e.g. real projective plane.

Divergence theorem

Suppose S is closed and orientable. Let E be the 3D region inside S. Suppose that $d\vec{S}$ is chosen to point **outward**. Let $\vec{F}$ be a vector field.

Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

(flux integral)

Rmk Because is closed, sometimes use the notation

$$\oiint_S \vec{F} \cdot d\vec{S}$$

to indicate that S is closed.

cf. $\oint_C \vec{F} \cdot d\vec{r}$

for line integral around closed curve.

Compare with Stokes.

S.T. $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \mathrm{curl}\,\vec{F} \cdot d\vec{S}$

D.T. $\oiint_S \vec{F} \cdot d\vec{S} = \iiint_E \mathrm{div}\,\vec{F}\, dV$.

LHS: integral of v.f. over a closed shape

RHS: higher integral interior and v.f. gets replaced by "derivative"

$\vec{F} = \langle P, Q, R \rangle$

$\text{div}\,\vec{F} = P_x + Q_y + R_z$

$= \nabla \cdot \vec{F}$

where $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$

curl $\Longleftrightarrow$ $\times$

div $\Longleftrightarrow$ $\cdot$

$\text{div}\,\vec{F}$ is a function in 3D.

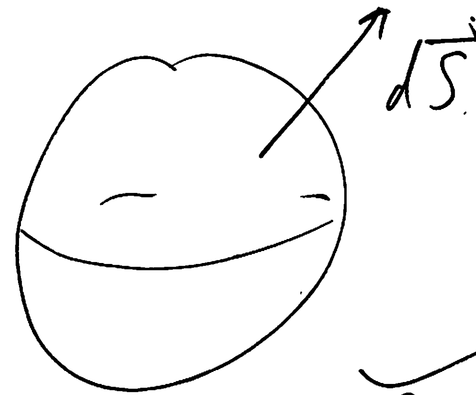
e.g. S = unit sphere.

$\vec{F} = \frac{1}{3}\langle x, y, z \rangle$

compute both sides

LHS: $\displaystyle\oiint_S \vec{F} \cdot d\vec{S}$

want outward normal.



on unit sphere, $\underbrace{x^2+y^2+z^2}_{f} = 1$

$\nabla f = \perp$ sphere $\langle 2x, 2y, 2z \rangle$

points in correct direction at every pt on S, but not $d\vec{S}$

$d\vec{S} = \hat{n}\, dS$

$\hat{n} = \langle x, y, z \rangle$ $\leftarrow$ unit vector $\perp$ to S.

$d\vec{S} = \langle x, y, z \rangle \, dS$

area element on S.

$\vec{F} = \frac{1}{3} \langle x, y, z \rangle$

$\vec{F} \cdot d\vec{S} =$

$\frac{1}{3} \langle x, y, z \rangle \cdot \langle x, y, z \rangle \, dS$

$= \frac{1}{3} (x^2 + y^2 + z^2) \, dS$

$= \frac{1}{3} dS$

$\oint\!\!\oint_S \vec{F} \cdot d\vec{S} = \frac{1}{3} \oint\!\!\oint_S dS$

$= \frac{1}{3} \text{area}(S)$

$= \boxed{\frac{4\pi}{3}} = LHS.$

RHS: $\iiint_E \operatorname{div} \vec{F} \, dV.$

$\vec{F} = \frac{1}{3} \langle x, y, z \rangle$

$\operatorname{div} \vec{F} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$

$\iiint_E 1 \, dV = \text{vol}(E).$

$= \boxed{\frac{4\pi}{3}}!$

$LHS = RHS$ ✓