Divergence Thm (Gauss's Thm)

$\vec{F}$ vector field (3D)

$= \langle P, Q, R \rangle$

$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$

S closed orientable surface with outward pointing normal

i.e. $d\vec{S}$ points outward.

E interior of S.

Divergence Thm

$$\oint\!\!\oint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

e.g. $\vec{F} = \langle \sin \pi x, zy^3, z^2 + 4x \rangle$ ①

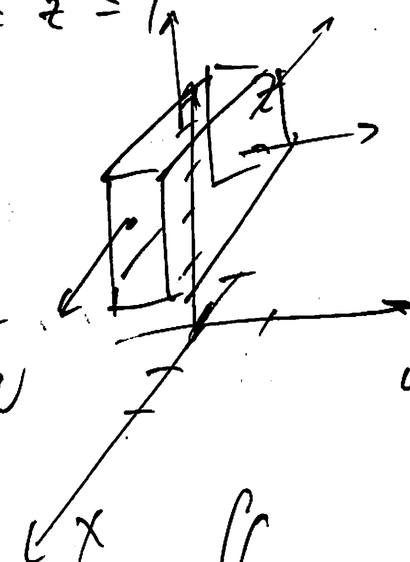
$S =$ box determined by

$-1 \le x \le 2$

$0 \le y \le 1$

$1 \le z \le 4$

want $\oint\!\!\oint_S \vec{F} \cdot d\vec{S}$



$S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$

6 sides of box

$$\oint\!\!\oint_S \cdots = \sum_{i=1}^{6} \iint_{S_i} \cdots$$

use Divergence Thm

$$\operatorname{div}\vec{F} = \pi\cos\pi x + 3y^2z + 2z$$

E :

$$\iiint_E \operatorname{div}\vec{F}\,dV = \int_{-1}^{2}\int_{0}^{1}\int_{1}^{4}\left(\pi\cos\pi x + 3y^2z + 2z\right)dz\,dy\,dx$$

... (pretty std triple integral)

$$= \boxed{\frac{135}{2}}$$

e.g. $\vec{F} = \langle xy, -\frac{1}{2}y^2, z\rangle$ 2

$S =$ graph of $z = 4-3x^2-3y^2$

$1 \le z \le 4$

+ cylinder $x^2+y^2=1$

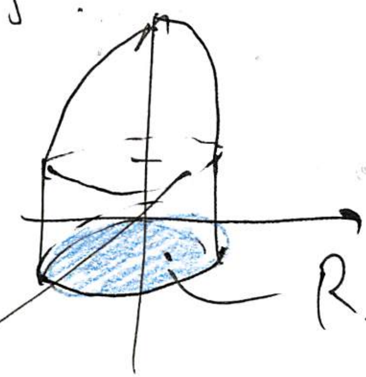
$0 \le z \le 1$

+ unit disk $x^2+y^2 \le 1$

$z=0$

outward normal.

want $\displaystyle\oiint_S \vec{F}\cdot d\vec{S}$.



compute $\displaystyle\iiint_E \operatorname{div}\vec{F}\,dV$ instead.

E = inside

$\operatorname{div}\vec{F} = y - y + 1 = 1$

$$\iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E dV = \text{volume.}$$

$$Vol = \iint_R (\text{height function})\, dA$$

R unit disk in xy plane

$$= \iint_R (4 - 3x^2 - 3y^2)\, dA$$

use polar.

$dA = r\,dr\,d\theta.$

R: $0 \le r \le 1$
$0 \le \theta \le 2\pi$

$r^2 = x^2 + y^2$

$\Rightarrow$ height fn is $4 - 3r^2$.

$$\int_0^1 \int_0^{2\pi} (4 - 3r^2)\, r\, d\theta\, dr$$

$$= 2\pi \int_0^1 (4r - 3r^3)\, dr$$

$$= 2\pi \left(2r^2 - \frac{3}{4}r^4\right)\Big|_0^1$$

$$= 2\pi \left(2 - \frac{3}{4}\right)$$

$$= \frac{10\pi}{4} = \boxed{\frac{5\pi}{2}}$$

e.g. suppose $\vec{F}$ itself is the curl of a vector field $\vec{G}$:

$$\vec{F} = \text{curl } \vec{G}.$$

claim

$$\iint_S \vec{F} \cdot d\vec{S} = 0$$

for all closed orientable surfaces S.

2 ways to prove this.

(1) divergence theorem.

E = interior of S

$$\begin{array}{ccc} \displaystyle\iint_S \vec{F} \cdot d\vec{S} & = & \displaystyle\iiint_E \text{div } \vec{F} \, dV. \\ \| & & \| \\ \displaystyle\iint_S \text{curl } \vec{G} \cdot d\vec{S} & & \displaystyle\iiint_E \text{div}(\text{curl } \vec{G}) \, dV \end{array}$$

But: $\text{div}(\text{curl } \vec{F}) = 0$.

recall: $\mathcal{F}$ = functions in 3D.

$\mathcal{H}$ = vector fields in 3D

Have diagram.

$$\mathcal{F} \xrightarrow{\text{grad.}} \mathfrak{X} \xrightarrow{\text{curl}} \mathfrak{X} \xrightarrow{\text{div.}} \mathcal{F}$$

The composition of 2 of these in order always gives 0.

(a) $\text{curl}(\text{grad} f) = \vec{0}$

(b) $\text{div}(\text{curl}\,\vec{G}) = 0$

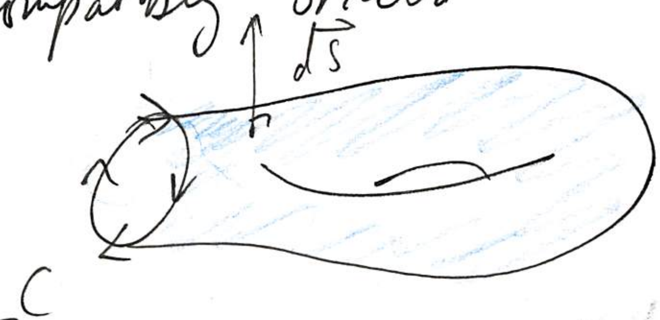
$$\Rightarrow \iiint_E \text{div}(\text{curl}\,\vec{G})\,dV = \iiint_E 0\,dV$$

$= 0$.

(T)

(2) can also use Stokes's Thm.

Let S be surface, orientable, with boundary curve C. assume S, C compatibly oriented

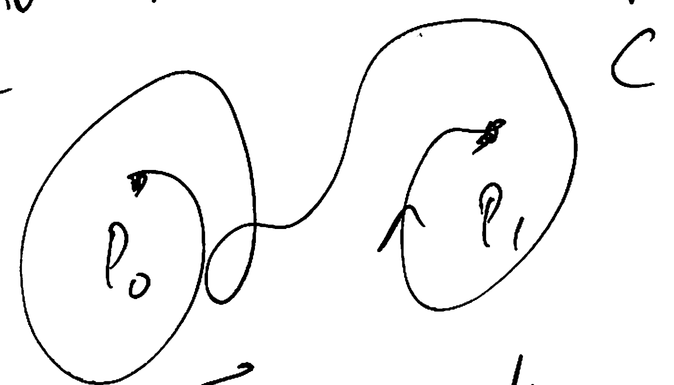


$$\boxed{\text{S.T.}}\ \oint_C \vec{G}\cdot d\vec{r} = \iint_S \text{curl}\, G \cdot d\vec{S}$$

For our example, S is <u>closed</u>. So it has <u>no</u> boundary curve.

$$\oiint_S \operatorname{curl} \vec{G} \cdot d\vec{S} = \oint_C \vec{G} \cdot d\vec{r} = 0$$

Recall that a v.f. $\vec{F}$ is conservative if $\int_C \vec{F} \cdot d\vec{r}$ only depends on the endpoints of C.



If $\vec{F}$ conservative, then we can find a potential function φ s.t.

$$\vec{F} = \operatorname{grad} \varphi.$$

Fundamental thm for line integrals

$$\int_C \text{grad}\,\varphi \cdot d\vec{r} = \varphi(P_1) - \varphi(P_0)$$

If C is closed, then

$$\oint_C \vec{F} \cdot d\vec{r} = 0.$$

$$\oint_C \text{grad}\,\varphi \cdot d\vec{r} = 0.$$

compare: if S is a closed orientable surface, then

$$\oiint_S (\text{curl}\,\vec{G}) \cdot d\vec{S} = 0$$

If $\vec{F} = \text{curl}\,\vec{G}$,

we say $\vec{G}$ is a <u>vector</u> <u>potential</u> for $\vec{F}$.

φ scalar potential.
grad φ gives desired v.f.

$\vec{G}$ vector potential
curl $\vec{G}$ gives desired v.f.