DISCLAIMER: This practice exam is intended to give you an idea about what a two-hour final exam is like. It is not possible for any practice exam to cover every topic, and the content, coverage and format of your actual exam could be different from this practice exam.

Part I: Multiple Choice Problems. You only need to give the answer; no justification needed.

1. Describe the surface whose equation in cylindrical coordinates is $z = 4r$.

(a) cylinder with vertical axis (b) cylinder with horizontal axis (c) sphere

(d) half cone with vertical axis (e) half cone with horizontal axis (f) plane

ANS: d SOLUTION: Observe that the level sets $z = c$ of this surface are given by $r = \frac{c}{4}$, i.e. they are circles centered about the z-axis. From this analysis, it is clear that the surface is a (half-)cone with vertical axis. Another way to see this is to change to spherical coordinates:

$$\begin{aligned} z = 4r \implies \rho\cos(\phi) &= 4\rho\sin(\phi) \\ \cot(\phi) &= 4 \text{ (when } \rho \neq 0) \\ \phi &= \operatorname{arccot}(4) \end{aligned}$$

and use that (half-)cones with apex at the origin are specified uniquely by $0 < \phi < \pi$.

2. Which of the following double integrals are such that reversing the order of integration would result in two double integrals?

A: $\int_{-1}^{2}\int_{x^2-2}^{x} f(x,y)\,dy\,dx$, **B:** $\int_{0}^{1}\int_{y^2}^{2-y} g(x,y)\,dx\,dy$, **C:** $\int_{0}^{1}\int_{\arctan x}^{\pi/4} h(x,y)\,dy\,dx$

(a) A only (b) B only (c) C only (d) A & B (e) A & C (f) B & C

ANS: d

SOLUTION: Interpret the question like so: which of the specified regions of integration can not be described in the opposite type? Drawing graphs will help.

A. The region of integration is $R = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 2,\ x^2-2 < y < x\}$, and it is Type I. Since there are distinct x-values in R which produce the same y-value on its lower bound (e.g. $(-1)^2 - 2 = (1)^2 - 2$), this bound fails a "horizontal line test" and is not invertible. So R can not be described in Type II using a single region.

B. The region of integration is $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1,\ y^2 \leq x \leq 2-y\}$, and it is Type II. The bounds $x = y^2$ and $x = 2 - y$ intersect at $x = 1$, which lies in the middle of the range of all x-values in R, namely $0 \leq x \leq 2$. Then the upper bound function for y changes within R and so R can not be described in Type I using a single region.

C. The region of integration is $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ \arctan(x) \leq y \leq \frac{\pi}{4}\}$, and it is Type I. One can quickly verify that R can be described in Type II as $\{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \frac{\pi}{4},\ 0 \leq x \leq \tan(y)\}$.

3. Which of the following points is a local maximum of the function $f(x,y) = xy - x^2y - xy^2$?

(a) $(0,0)$ (b) $(1,1)$ (c) $(0,2)$ (d) $(2,0)$ (e) $(1/3,1/3)$ (f) $(1/2,1/2)$

ANS: e

SOLUTION: f is differentiable everywhere, so its critical points are when $\nabla f = \vec{0}$:

$$\begin{aligned}\nabla f &= \langle y - 2xy - y^2,\ x - x^2 - 2xy\rangle \\ &= \langle y(1-2x-y),\ x(1-x-2y)\rangle\end{aligned}$$

The first component vanishes when $y = 0$ or $y = 1 - 2x$ and the second component vanishes when $x = 0$ or $x = 1-2y$. Assuming $y = 0$, we get the critical points $(0,0)$ and $(1,0)$. Assuming $y = 1 - 2x$, we get the critical points $(0,1)$ and $(\frac{1}{3}, \frac{1}{3})$. Only $(0,0)$ and $(\frac{1}{3}, \frac{1}{3})$ are among our options, so we can consider the determinant of the Hessian matrix evaluated at just these points.

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix} = \begin{vmatrix} -2y & 1-2x-2y \\ 1-2x-2y & -2x\end{vmatrix} = 4xy - (1-2x-2y)^2.$$

$D(0,0) = -1$, so the origin must be a saddle point. By process of elimination, it must be that $(\frac{1}{3}, \frac{1}{3})$ is a local maximum. Indeed, $D(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3}$ and $f_{xx}(\frac{1}{3}, \frac{1}{3}) = -2(\frac{1}{3}) < 0$.

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4. Determine the set on which the function $f(t) = \dfrac{\sqrt{t}-1}{\sqrt{t}+1}$ is continuous.

(a) $t > 0$ (b) $t \geq 0$ (c) t >1 (d) $t \geq 0$ and $t \neq 1$ (e) $t \neq 0$ (f) all real numbers

ANS: b

SOLUTION: $f(t)$ is fractional, so we must exclude any points where the denominator is 0. Furthermore, it contains $\sqrt{t}$, which is only defined in $\mathbb{R}$ when $t \geq 0$. The denominator is never zero since $\sqrt{t} + 1 \geq 1$, so the domain of $f(t)$ is all of $t \geq 0$. Since $f(t)$ is defined in one piece on its domain with no removable discontinuities, it follows that $f(t)$ is continuous on its entire domain. (It might be tempting to say there is a discontinuity at $t = 0$ since no left limit exists, but the notion of left limit itself is not well-defined here since there are no values $t < 0$ in the domain to take the limit at.)

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5. Let $f(x,y,z)$ be a differentiable function, and let $\vec{F}(x,y,z)$ be a differentiable, 3-dimensional vector field. Which of the following formulae is correct?

(a) $\text{div}(f\vec{F}) = f\text{div}\vec{F} + \text{curl}\vec{F} \bullet \nabla f$ (b) $\text{div}(f\vec{F}) = f\text{curl}\vec{F} + \vec{F} \times \nabla f$

(c) $\text{curl}(f\vec{F}) = f\text{div}\vec{F} + \vec{F} \bullet \nabla f$ (d) $\text{div}(f\vec{F}) = f\text{div}\vec{F} + \vec{F} \bullet \nabla f$

(e) $\text{curl}(f\vec{F}) = f\text{curl}\vec{F} + \text{curl}\vec{F} \times \nabla f$ (f) none of the above

ANS: d

SOLUTION: The formula (d) is a direct consequence of the product rule:

$$\begin{aligned}\text{div}(f\vec{F}) &= \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR) \\ &= f\frac{\partial P}{\partial x} + P\frac{\partial f}{\partial x} + f\frac{\partial Q}{\partial y} + Q\frac{\partial f}{\partial y} + f\frac{\partial R}{\partial z} + R\frac{\partial f}{\partial z} \\ &= f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) + P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} \\ &= f\text{div}(\vec{F}) + \vec{F} \bullet \nabla f\end{aligned}$$

The others options are either not well-defined or not correct.

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6. Let $\vec{v} \in \mathbb{R}^3$ be a constant, non-zero vector. Denote by S the surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$ with inward-pointing normal vectors. Compute the surface integral $\iint_S 2\vec{v} \bullet \vec{n} dS$.

   (a) $-2|\vec{v}|$ (b) $-|\vec{v}|$ (c) 0 (d) $|\vec{v}|$ (e) $2|\vec{v}|$ (f) $3|\vec{v}|$

ANS: c

SOLUTION: Say that $\vec{v} = \langle a, b, c\rangle$ and the faces of S are S_1 through S_6. The unit normal vectors for each pair of faces of the cube are $\langle \mp 1, 0, 0\rangle$, $\langle 0, \mp 1, 0\rangle$ and $\langle 0, 0, \mp 1\rangle$. Then

$$\begin{aligned}\iint_S 2\vec{v} \bullet \vec{n}\, dS &= 2\iint_S \vec{v} \bullet \vec{n}\, dS \\ &= 2\left(\iint_{S_1} \vec{v} \bullet \langle -1, 0, 0\rangle\, dS + \iint_{S_2} \vec{v} \bullet \langle 1, 0, 0\rangle\, dS + \cdots\right) \\ &= 2\left(-a\iint_{S_1} dS + a\iint_{S_2} dS + \cdots\right)\end{aligned}$$

Since S_1 and S_2 have the same area, we have $\iint_{S_1} dS = \iint_{S_2} dS = B$ and get $(-a + a)B = 0$ in the parentheses. The same kind of cancellation will happen with the other pairs of faces.

PART II: Written Problems. To earn full credit for the following problems you must show your work. You can leave answers in terms of fractions and square roots.

1. Find the point at which the two lines
$$\vec{r_1}(t) = \langle 1,1,0\rangle + t\langle 1,-1,2\rangle, \qquad \vec{r_2}(s) = \langle 2,0,2\rangle + s\langle -1,1,0\rangle$$
intersect.

ANS: $(\mathbf{2},\mathbf{0},\mathbf{2})$

SOLUTION: The two lines intersect only if all three components are equal. As such, we must solve the system of equations

$$\begin{aligned} 1+t &= 2-s \\ 1-t &= s \\ 2t &= 2 \end{aligned}$$

Clearly $t = 1$, then from the second equation we get $s = 0$, and we can verify that the first equation is consistent: $1 + (1) = 2 - (0)$ indeed. So the intersection point is $\vec{r_1}(1) = \vec{r_2}(0) = (2, 0, 2)$.

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2. Find the work done by the vector field $\vec{F} = x^2y\vec{i} + \frac{1}{3}x^3\vec{j} + xy\vec{k}$ along the curve of intersection of the paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

ANS: π Joules

SOLUTION: First, let us parametrize the curve C. Since it lies on a cylinder, the natural parameter to use is θ. Then $\vec{r}(\theta) = \langle\cos(\theta), \sin(\theta), \sin^2(\theta) - \cos^2(\theta)\rangle$ for $0 \le \theta \le 2\pi$ describes C. Next, we compute $\vec{r}'(\theta) = \langle -\sin(\theta), \cos(\theta), 4\cos(\theta)\sin(\theta)\rangle$. Now consider the vector field restricted to C, given by

$$\vec{F}(\vec{r}(\theta)) = \langle\cos^2(\theta)\sin(\theta), \frac{1}{3}\cos^3(\theta), \cos(\theta)\sin(\theta)\rangle.$$

Dotting this with $\vec{r}'(\theta)$ produces a scalar field we can integrate over to calculate work done by $\vec{F}$. That is, we have:

$$\begin{aligned} \int_C \vec{F} \bullet d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(\theta)) \bullet \vec{r}'(\theta)\, d\theta \\ &= \int_0^{2\pi} 3\cos^2(\theta)\sin^2(\theta) + \frac{1}{3}\cos^4(\theta)\, d\theta \\ &= \int_0^{2\pi} \cos^2(\theta)\left(3\sin^2(\theta) + \frac{1}{3}\cos^2(\theta)\right) d\theta \\ &= \int_0^{2\pi} \cos^2(\theta)\left(3 - \frac{8}{3}\cos^2(\theta)\right) d\theta \\ &= \int_0^{2\pi} \left(\frac{\cos(2\theta)+1}{2}\right)\left(\frac{5-4\cos(2\theta)}{3}\right) d\theta \\ &= \frac{1}{6}\int_0^{2\pi} \left(5 + \cos(2\theta) - 4\cos^2(2\theta)\right) d\theta \\ &= \frac{1}{6}\int_0^{2\pi} \left(3 + \cos(2\theta) - 2\cos(4\theta)\right)\, d\theta = \frac{1}{2}\int_0^{2\pi}\, d\theta = \pi. \end{aligned}$$

(At the end, we used the fact we are integrating over an integer multiple of a full period to simplify.) You could also try doing this with Stokes' theorem and not have to invoke the power reduction formulae if you're clever, but the rest of the set up and algebra will be more tedious.

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3. Find every point on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ at which the tangent plane is parallel to the plane $3x - y + 3z = 1$.

> **ANS:** $(\frac{-6}{5\sqrt{2}}, \frac{1}{5\sqrt{2}}, \frac{-2}{5\sqrt{2}})$, $(\frac{6}{5\sqrt{2}}, \frac{-1}{5\sqrt{2}}, \frac{2}{5\sqrt{2}})$
>
> **SOLUTION:** First, we note that the normal vector to the plane is $\vec{n} = \langle 3, -1, 3\rangle$, and any other plane parallel to the given one must have a scalar multiple of $\vec{n}$ as its normal vector. We can determine the normal vectors for tangent planes to a surface defined by a level set $f(x, y, z) = c$ by calculating ∇f. Here $f = x^2 + 2y^2 + 3z^2$ so $\nabla f = \langle 2x, 4y, 6z\rangle$ gives the desired normal vector at a point (x, y, z). Now we can solve a system of four equations in four unknowns:
>
> $$3\lambda = 2x,\ -\lambda = 4y,\ 3\lambda = 6z,\ x^2 + 2y^2 + 3z^2 = 1$$
>
> The first and third equations tell us $x = 3z$; the second and third tell us $z = -2y$, which then also means $x = -6y$. Now we can solve the last equation in just y:
>
> $$(-6y)^2 + 2y^2 + 3(-2y)^2 = 1$$
> $$36y^2 + 2y^2 + 12y^2 = 1 \implies y^2 = \frac{1}{50}$$
>
> So then the two points on the ellipsoid satisfying our condition are $(\frac{\mp 6}{5\sqrt{2}}, \frac{\pm 1}{5\sqrt{2}}, \frac{\mp 2}{5\sqrt{2}})$.

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4. Consider the vector field $\vec{F}(x, y, z) = y\vec{i} + (x + z)\vec{j} + y\vec{k}$.
   (a) Is this vector field conservative? If so, find a potential function for $\vec{F}$; if not, explain.
   (b) Compute the line integral of $\vec{F}$ along the line segment from $(2, 1, 4)$ to $(8, 3, -1)$.

> (a) Is this vector field conservative? **ANS: Yes, potential function is $\mathbf{y(x + z) + c}$**
>
> **SOLUTION:** Let us check if there exists a potential function:
>
> - $\int y\, dx = xy + f(y, z)$
> - $\int (x + z)\, dy = xy + yz + g(x, z)$
> - $\int y\, dz = yz + h(x, y)$
>
> The first and second antiderivatives both have xy, and we can include it in $h(x, y)$ for the third. The second and third antiderivatives both have yz, and we can include it in $f(y, z)$ for the first. All terms are accounted for, so then $\vec{F} = \nabla f$ with $f(x, y, z) = xy + yz = y(x+z) + C$.
>
> (b) Compute the line integral of $\vec{F}$... **ANS: 15**
>
> **SOLUTION:** Since $\vec{F}$ is conservative, we can just use the fundamental theorem:
>
> $$\int_C \vec{F} \bullet d\vec{r} = f(b) - f(a)$$
>
> where a, b are the respective endpoints of C. In our case: $f(x, y, z) = y(x + z)$, $a = (2, 1, 4)$ and $b = (8, 3, -1)$. So then $f(8, 3, -1) - f(2, 1, 4) = 3(8 + (-1)) - 1(2 + 4) = 21 - 6 = 15$.

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5. Evaluate $\iint_S \vec{F} \bullet d\vec{s}$ where $\vec{F} = \langle -x, -y, z^3\rangle$ and S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$, with downward orientation.

ANS: $\frac{-1712\pi}{15}$

SOLUTION: First, we can parametrize the surface naturally as $\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2}\rangle$. Then we calculate

$$\vec{r}_x = \langle 1, 0, \frac{x}{\sqrt{x^2+y^2}}\rangle$$
$$\vec{r}_y = \langle 0, 1, \frac{y}{\sqrt{x^2+y^2}}\rangle$$
$$\vec{r}_x \times \vec{r}_y = \langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\rangle$$

Since the z-coordinate is positive, this is actually the inward normal vector pointing *toward* the cone, so we just take $-(\vec{r}_x \times \vec{r}_y) = \vec{r}_y \times \vec{r}_x$ instead once we integrate:

$$\iint_S \vec{F} \bullet d\vec{r} = \iint_R \langle -x, -y, (x^2+y^2)^{\frac{3}{2}}\rangle \bullet \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\rangle \, dA$$
$$= -\iint_R \sqrt{x^2+y^2} + (x^2+y^2)^{\frac{3}{2}} \, dA$$

The natural way to evaluate this is to use polar coordinates, since the cone is described here as $z = r$, we get

$$-\iint_R \sqrt{x^2+y^2} + (x^2+y^2)^{\frac{3}{2}} \, dA = -\int_0^{2\pi} \int_1^3 (r + r^3) \, r dr d\theta$$
$$= -2\pi \int_1^3 r^2 + r^4 \, dr$$
$$= -2\pi \left[\frac{1}{3}r^3 + \frac{1}{5}r^5\right]_1^3$$
$$= -2\pi \left(9 + \frac{243}{5} - \frac{1}{3} - \frac{1}{5}\right)$$
$$= -\frac{1712\pi}{15}$$

Another way you could calculate this is to take a surface integral over the given part of the cone *along with* the top and bottom discs contained in $z = 3$ and $z = 1$ respectively, then invoke Divergence Theorem. After that, just calculate the flux through the top and bottom discs directly and subtract those contributions from the Divergence Theorem result. Here's the first piece required using this method, see if you understand why and finish off the rest:

$$\int_0^{2\pi} \int_1^3 \int_0^z (3z^2 - 2) \, r dr dz d\theta$$