MATH 233H FINAL (TAKE HOME)

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and *any four* of the remaining problems. You must justify your answer to receive credit for a solution; correct answers alone are not necessarily sufficient for credit.

The exam will be submitted on Moodle as individual problems. Please submit exactly five problems (including Problem 1); if you submit more than five (including submitting Problem 1) only the first five (in numerical order) will be graded. If you submit more than four and don't submit Problem 1, then only the first four in numerical order will be graded.

Please make sure your name and student ID are written somewhere in your answers. PDF must be submitted on Moodle; no other formats (such as doc, jpg, tiff, etc) will be accepted. Also please name your PDF files in the form

`StudentID_LastName_FirstName_Exam1_ProblemX.pdf`

where X is the problem number. For example,

`314159_Gunnells_Paul_Exam1_Problem1.pdf`

ADDITIONAL INSTRUCTIONS FOR TAKE-HOME EXAM.

The exam answers must be submitted in PDF. Scans of handwriting are ok, but please be sure that they are at a sufficiently high resolution for me to be able to read them. The following **are allowed**:

- You may use class materials (textbook, your own notes, hw assignments, lecture notes from video lectures, video lectures, and other materials on our course pages) during the exam.
- You may use the Desmos Scientific calculator `https://www.desmos.com/scientific` to assist with numerical computations. Algebraic computations must be done by hand. You may also use your own calculator if you prefer it; Desmos is allowed so that everyone is guaranteed to have access to something.

The following **are not allowed**:

- Discussing the exam with anyone in the class or elsewhere. Exception: you may ask me by email for clarification about a problem, just like in the classroom exam. I will try to check email often but unavoidably there will be delays in replies.
- Using any other sources of information (internet, other books, other notes, tables, Wikipedia, etc.) during the exam. In particular you are allowed to look at your own HW, but not any materials away from WebAssign.
- Using a computer (other than Desmos above or for access to video lectures and our course page). In particular programming is not allowed.

When submitting your exam, you are agreeing to the following statement:

I hereby declare that the work submitted represents my individual effort. I have neither given nor received any help and have not consulted any online resources other than those authorized. I attest that I have followed the instructions of the exam.

Academic honesty is very important to me.

Date: Assigned: Monday 30 Nov 2020; Due: on Moodle by 6pm, Thursday 22 Oct 2020.

Let me know if you find any mistakes in the answers.

(1) (20 points) Please compute the following. In this problem (and only this problem), there is no partial credit awarded and it is sufficient to just write the answers of the computations.

(a) (4 points) Let $\vec{\boldsymbol{v}} = \langle 1, 2, -1\rangle$ and $\vec{\boldsymbol{w}} = \langle 0, 1, -2\rangle$. Compute $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}$ and $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}$. **Answer:** $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} = 4$ and $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}} = \langle -3, 2, 1\rangle$.

(b) (4 points) Let $P = (0, 1, 1)$, $Q = (1, -1, 0)$, $R = (1, 0, 3)$. Find the area of the triangle with vertices at P, Q, R. **Answer:** We use $|QP \times RP|/2 = |\langle 1, -2, -1\rangle \times \langle 1, -1, 2\rangle|/2 = \sqrt{35}/2$

(c) (4 points) Let $g(x, y) = \cos(xy) + x^2y + y^3$. Find an equation for the tangent plane to the graph of g at $(0, 0, 1)$. **Answer:** The normal vector is $\langle -g_x, -g_y, 1\rangle$ at $(0, 0, 1)$, and this is $\langle 0, 0, 1\rangle$. So the equation is $z = 1$.

(d) (4 points) Let $\vec{\boldsymbol{F}}(x, y, z) = \langle x^2 + y, z - x, x^2 - z^2\rangle$. Compute $\text{curl}\vec{\boldsymbol{F}}$ and $\text{div}\vec{\boldsymbol{F}}$. **Answer:** $\text{curl}\vec{\boldsymbol{F}} = \langle -1, -2x, -2\rangle$ and $\text{div}\vec{\boldsymbol{F}} = 2x - 2z$.

(e) (4 points) Compute $\oint_C x\,dy$ where C is the square running from $(0, 0)$ to $(1, 0)$ to $(1, 1)$ to $(0, 1)$ and then back to $(0, 0)$. **Answer:** Using Green's theorem we see that this is just the area of the square, which is 1.

(2) (20 points) Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 12$. **Answer:** The volume function is $V = xyz$ and we must have $x, y, z \geq 0$. There are two ways to proceed: (i) we can use the use the equation for the plane to solve for one of the variables in terms of the other two, which leads to a "max/min on a closed domain" problem, or (ii) we can use Lagrange multipliers, which is actually easier. If we do Lagrange, then the system is $\lambda = yz$, $2\lambda = xz$, $3\lambda = xy$, $x + 2y + 3z = 12$. This is not difficult to solve. We must have $\lambda \neq 0$ because otherwise the volume is forced to be zero. Eliminating λwe get $x = 2y, x = 3z$. If we put these into the equation of the plane we get $3x = 12$ or $x = 4$. Then $y = 2$, $z = 4/3$, and the volume is $64/6 = 32/3$. This has to be a maximum because the solution is unique and positive, and because the volume is zero when any of the variables are 0.

(3) (20 points) Evaluate $\iiint_E xyz\,dV$, where E is the region between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\varphi = \pi/3$. **Answer:** We use spherical coordinates. After conversion the integral is

$$\int_2^4 \int_0^{\pi/3} \int_0^{2\pi} \rho^5 \sin^3\varphi \cos\varphi \sin\theta \cos\theta \, d\theta \, d\varphi \, d\rho.$$

The integral with respect to θ gives 0, so the whole integral evaluates to 0.

(4) (20 points) Let $\alpha > 0$ be a constant and let S be the graph of $z = \sqrt{\alpha x^2 + \alpha y^2}$ between $z = 0$ and $z = \sqrt{\alpha}$. Compute the surface area of S. **Answer:** The surface sits over the base region R given by the disk in the xy-plane of radius 1. We will eventually use polar, since the surface is symmetric under rotation about the z-axis, and this region R becomes $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$. We need to integrate $\sqrt{1 + z_x^2 + z_y^2}$ over R. This is computed to be $\sqrt{1+\alpha}$. Thus we compute

$$\int_0^{2\pi} \int_0^1 \sqrt{1+\alpha}\, dA = (\text{area of } R) \cdot \sqrt{1+\alpha} = \pi\sqrt{1+\alpha}.$$

(5) (20 points) Let R be the region in the xy plane between the graphs of $xy = 2$, $xy = 6$, $y = 4$ and $y = 10$. Use the change of variables $x = v$, $y = (2u)/(3v)$ to compute

$$\iint_R \frac{15y}{x}\, dA.$$

Answer: After changing variables the region becomes $3 \le u \le 9$ and $u/15 \le v \le u/6$. The Jacobian is $-2/(3v)$, so we have to take the absolute value to get $2/(3v)$. The integral becomes

$$\int_3^9 \int_{u/15}^{u/6} \frac{20u}{3v^3}\, du\, dv = 630\ln(3).$$

(6) (20 points) Let $\vec{\boldsymbol{F}}$ be the vector field $\langle 3x^2e^{2y} + 4ye^{4x}, 1 + 2x^3e^{2y} + e^{4x}\rangle$. Compute $\int_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$, where C is the curve along the parabola $y = x^2$ going from the origin to $(1, 1)$. **Answer:** This vector field is conservative; a potential is given by $\varphi(x, y) = x^3e^{2y} + ye^{4x} + y$. Using the fundamental theorem of line integrals the answer is $\varphi(1, 1) - \varphi(0, 0) = e^4 + e^2 + 1$.

(7) (20 points) Let $\vec{\boldsymbol{F}}$ be the vector field $\langle x^3z - 2z, xz, xy\rangle$. Let C the closed triangular curve running along line segments from $(0, 0, 4)$ to $(0, 2, 0)$ to $(2, 0, 0)$ and back to $(0, 0, 4)$. Compute

$$\oint_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}.$$

Answer: We use Stoke's theorem and compute $\iint_S \mathrm{curl}\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{S}}$ where S in the interior of the triangle. First $\mathrm{curl}\vec{\boldsymbol{F}} = \langle 0, x^3 - y - 2, z\rangle$. Since S is a graph $z = 4 - 2x - 2y$ over the base region R given by $0 \le x \le 2$, $0 \le y \le 2 - x$ we can take $d\vec{\boldsymbol{S}}$ to be one of $\pm\langle -z_x, -z_y, 1\rangle\, dx\, dy$ (we have to pick the sign according to the orientation of C.) Based on the orientation of C we need $d\vec{\boldsymbol{S}}$ to point down (negative z component), so we take $d\vec{\boldsymbol{S}} = \langle -2, -2, -1\rangle$. The integral we have to compute is

$$\iint_S \mathrm{curl}\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{S}} = \int_0^2 \int_0^{2-x} (-2x^3 + 4y + 2x)\, dy\, dx = 24/5.$$

(8) (20 points) Let $\beta > 0$ be a constant and let P be the octahedral surface with vertices $(\beta, 0, 0)$, $(-\beta, 0, 0)$, $(0, \beta, 0)$, $(0, -\beta, 0)$, $(0, 0, \beta)$, $(0, 0, -\beta)$. (see Figure 1.) Assume that P is oriented with the outward pointing normal. Let $\vec{\boldsymbol{F}}(x, y, z) = \langle x - y - z, -x + y - z, -x - y + z\rangle$. Compute the flux of $\vec{\boldsymbol{F}}$ through P:

$$\oiint_P \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{S}}.$$

Answer: We use the divergence theorem and integrate $\mathrm{div}\vec{\boldsymbol{F}}$ over the inside E of P. Since $\mathrm{div}\vec{\boldsymbol{F}} = 3$, the integral is 3 times the volume of E. The region E is cut into 8 congruent tetrahedra by the octants. The one in the first octant is similar to one that we've done before: if we take the vertices to be the origin and $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, then the volume is $1/6$. We can either mimic that computation using our points or can observe that scaling the sides of the tetrahedron by β scales the volume by β^3. Thus the volume of this tetrahedron is $\beta^3/6$. The volume of E times 3 is then $3 \cdot 8 \cdot \beta^3/6 = 4\beta^3$.

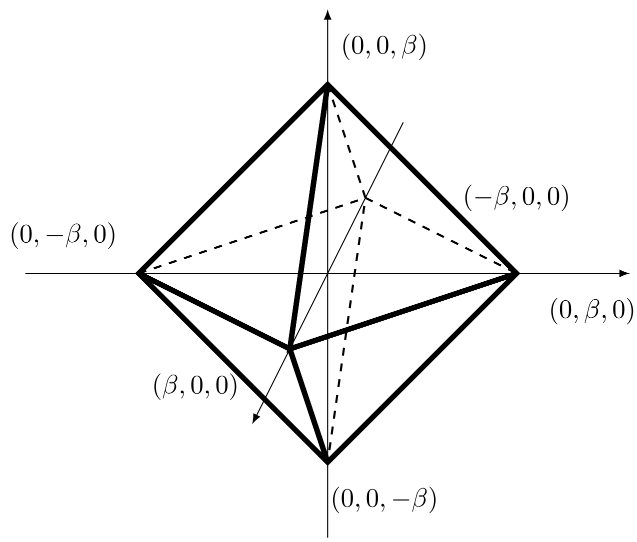

FIGURE 1. The octahedron P