MATH 513 EXAM I

This exam is worth 100 points, with each problem worth 20 points. There are problems on *both sides* of the page. Please complete Problems 1 and 2 and then *any three* of the remaining problems. Unless indicated, you must justify your answer to receive credit for a solution; correct answers alone are not necessarily sufficient for credit.

When submitting your exam, please indicate which problems (including Problems 1 and 2) you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly five problems; any unselected problems will not be graded, and if you select more than five only the first five (in numerical order) will be graded. You may use a calculator to assist with arithmetic. You can only use the basic functions (+, −, ×, /, =, memory) of the calculator; no advanced functions/programming is allowed. No phones, smartwatches, or other devices with connectivity can be used during the exam. The notation $[\![n]\!]$ means the finite set $\{1, \ldots, n\}$. Unless requested, answers may be left in symbolic form.

(1) (20 pts) **True/False.** Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.
   (a) (4 pts) Let X and Y be finite sets. A function $f\colon X \to Y$ is called *one to one* if for every $x \in X$, there is a $y \in Y$ such that $f(x) = y$.
   (b) (4 pts) The partition $(4, 2, 1, 1)$ of 8 is self-conjugate.
   (c) (4 pts) The binomial coefficients $\binom{n}{k}$ satisfy the recursion $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
   (d) (4 pts) Let A, B be finite subsets of a set C. Then we have $|A \cup B| = |A| + |B| - |A \cap B|$.
   (e) (4 pts) The partition function $p(n)$ satisfies the recursion $p(n) = p(n-1) + p(n-2)$.

(2) (20 pts) **Short Answer.** Please give the answer to these short computations.
   (a) (4 pts) Compute the binomial coefficient $\binom{7}{3}$.
   (b) (4 pts) Compute the Stirling number of the second kind (set Stirling number) $S(7, 3)$.
   (c) (4 pts) Compute the Stirling number of the first kind (cycle Stirling number) $s(7, 3)$.
   (d) (4 pts) Compute the number of *compositions* of 7 into 3 parts.
   (e) (4 pts) Compute $p_3(7)$, the number of *partitions* of 7 into at most 3 parts.

(3) (20 pts)
   (a) (10 pts) The door of an apartment building can only be opened by a 4-digit code, where digit means 0, 1, . . . , 9. A person forgot the code but remembers that it only uses both of the digits 3 and 5 and no other digits. How many possible codes are there?
   (b) (10 pts) After trying the codes from the first part, the person realized that actually the code uses the digits 3 and 5 and also 8, but no other digits. How many possible codes are there now?

Date: Tuesday, 14 Oct 2025.

(4) (20 pts)
  (a) (12 pts) In 2021 five friends run in a footrace every day for the full year (365 days). The races never end in a tie (meaning all racers finish at distinct times). Show that there must be at least two days where the races have the same outcome.
  (b) (8 pts) Suppose that in 2022 a sixth friend joins the group and they run a footrace every day for the full year, and again there are no ties. Can we conclude that there must be at least two days where the races have the same outcome?

(5) (20 pts) A small library surveyed 150 students about which kinds of books they read. The possible categories were Fantasy, Science Fiction, and Mystery. They recorded the following results: 40 students read Fantasy books. 35 students read Science Fiction books. 30 students read Mystery books. 15 students read both Fantasy and Science Fiction. 10 students read both Fantasy and Mystery. 8 students read both Science Fiction and Mystery. 5 students read all three kinds.
  (a) (6 pts) How many students surveyed read at least one of the three kinds?
  (b) (7 pts) How many students surveyed don't read any of these types of books?
  (c) (7 pts) How many only read Mystery books and don't read Fantasy or Science Fiction?

(6) (20 pts) There are four tip jars at a restaurant, one for each staff group. One is for the bartenders, one for the waitstaff, one for the hostess, and one for the piano player. A customer has 40 one-dollar bills and wants to tip. All \$40 gets used to tip.
  (a) (6 pts) How many ways are there to tip if every staff group gets at least one dollar?
  (b) (7 pts) How many ways are there to tip if the piano player gets nothing, but every other group gets something?
  (c) (7 pts) How many ways are there to tip if the bartenders, waitstaff, and hostess get the same amount and each group always get more than the piano player?

(7) (20 pts) A teacher has five students named Alice, Bob, Carol, Don, and Eve. They have to be assigned to groups for a group project. A group consisting of a single person is allowed unless said otherwise.
  (a) (5 pts) How many ways can they be assigned to groups if we don't care how many groups there are?
  (b) (5 pts) How many ways can they be assigned to groups if we only want two groups?
  (c) (5 pts) How many ways can they be assigned to groups if we only want two groups and single person groups are *not* ok?
  (d) (5 pts) How many ways can they be assigned to groups if Alice and Carol are always in the same group and Bob and Don are always *not* in the same group?

(8) (20 pts) Suppose we have a set $S = \{a_1, \dots, a_{10}\}$ of distinct positive integers. Suppose also that the largest integer in S is 20. Show that there are two different nonempty subsets A, B of S such that the sum of the numbers in A equals the sum of the numbers in B. ($A \neq B$ but $A \cap B \neq \emptyset$ is possible.)

(9) (20 pts) The chromatic polynomial of the 4-cycle C_4 (Figure 1) is $n^4 - 4n^3 + 6n^2 - 3n$. Verify this by using the method of inclusion/exclusion as demonstrated in class.

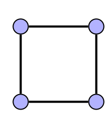

FIGURE 1. The 4-cycle C_4