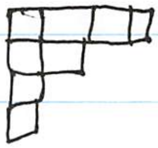
Math 513 / CS 575 Fall 2025 Exam 1 Answers

(1) (a) $\boxed{\text{False}}$. One to one says If $x \neq x'$ then $f(x) \neq f(x')$ (b) $\boxed{\text{True}}$, the diagram is



(c) $\boxed{\text{True}}$, this is the identity that gives the triangle

(d) $\boxed{\text{True}}$, this is I/E with 2 sets (also accepted False since it doesn't say finite sets)

(e) $\boxed{\text{False}}$, $p(3) = 3$ 3, 21, 111

$p(4) = 5$ 4, 31, 22, 211, 1111

$p(5) = 7$ 5, 41, 32, 311, 221, 2111, 11111

(2) (a) $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = \boxed{35}$

(b) use the recursion $S(n+1,k) = k\,S(n,k) + S(n,k-1)$

and build the triangle

k=1   k=2   k=3

1

0 1

0 1 1

0 1 3 1

0 1 7 6 1

0 1 15 25

0 1 31 90

n=7   0 1 63 $\boxed{301}$

(1) (c) use the recursion $c(n+1,k) = n\,c(n,k) + c(n,k-1)$

for the unsigned version then fix the sign

	1	$k=1$			
	0	1	$k=2$		
$n=2$	0	1	1	$k=3$	
$n=3$	0	2	3	1	
$n=4$	0	6	11	6	1
$n=5$	0	24	50	35	
$n=6$	0	120	274	225	
$n=7$	0	720	1764	$\boxed{1624}$	

Now $s(7,3) = (-1)^{7-3} c(7,3) = \boxed{1624}$

(d) comp. of 7 into 3 parts is same as weak comp. of $4 = 7-3$ into 3 parts. This is $\binom{4+3-1}{3-1} = \binom{6}{2} = \boxed{15}$

(e) writing them out, we find
7, 61, 52, 511, 43, 421, 331, 322
so $\boxed{8}$

(3) (a) There are $2^4 = 16$ length 4 sequences built from the digits 3, 5. Since 3, 5 both have to appear, we must eliminate 3333 and 5555, so $\boxed{14}$

(b) First there is a unique digit that is doubled in the sequence, and there are $\binom{4}{2} = 6$ ways to place the doubled digit. Then there are 2 ways to insert the remaining 2 digits. Since there are 3 choices for the doubled digit, we get $3 \cdot 6 \cdot 2 = \boxed{36}$ possibilities.

(4) (a) Use pigeonhole principle. The pigeons are the days of the year (365) and the pigeonholes are the number of possible race outcomes. There are $120 = 5!$ possible orderings of $[5]$, so thats how many outcomes there are. Since $365 > 120$ there must be at least one outcome that happens on 2 different days.

(4) (b) With 6 friends we have $6! = 720$ possible race outcomes. Since $365 \not> 720$ we can't conclude that there must be ≥ 2 days when the outcome is the same. Indeed we could have friend 1 finish 1st for the first 120 days, friend 2 finish first for the next 120, friend 3 for the next, and friend 4 for the last 5 days.

(5) We use I/E. Let the 3 main sets be F, S, and M. Then the orders of the relevant sets are

set	F	S	M	$F \cap S$	$F \cap M$	$S \cap M$	$F \cap S \cap M$
order	40	35	30	15	10	8	5

(a) we want $|F \cup S \cup M| = 40+35+30$
$-15-10-8+5$
$= \boxed{77}$

(b) we want $\text{Total} - |F \cup S \cup M| = 150 - 77$
$= 73$

5 (c) We want $|M - M \cap (F \cup S)|$. This will be $|M - X|$, where $X = (M \cap F) \cup (M \cap S)$, which will be $|M \cap F| + |M \cap S| - |M \cap F \cap S| = |X|$

So we want

$$30 - 10 - 8 + 5 = \boxed{17}$$

6 (a) We want the # of strong compositions of 40 into 4 pts, which is the same as the # of weak compositions of 36 into 4 pts. So we want

$$\binom{36+4-1}{4-1} = \binom{39}{3} = \boxed{9139}$$

(b) Now one jar is empty so only have 3 pts. 40 into 3 strong comp $\Longleftrightarrow$ 37 into 3 weak comp $= \binom{37+3-1}{3-1} =$

$$\binom{39}{2} = \boxed{741}$$

(c) $\underbrace{\_\ \_\ \_}_{a \text{ each}}\ \underset{b}{\_}$ and $3a+b = 40$, $a > b \geq 0$, $a, b \in \mathbb{Z}$.

Easiest just to enumerate solutions. Min value of b is 1, giving $a = 13$. b must ~~decrease~~ increase by 3 each time. So

6 (c) (cont'd) solutions are
$(13,1)$, $(12,4)$, $(11,7)$, $\cancel{(10,10)}$

$\Rightarrow$ $\boxed{3}$ possibilities

7 (a) all set partitions of $[\![5]\!] \Rightarrow$ Bell number $\boxed{B_5 = 52}$

(b) $\boxed{S(5,2) = 15}$ (set partitions of $[\![5]\!]$ into 2 parts)

(c) only possible config is $\{\bullet\bullet\bullet\}, \{\bullet\bullet\}$
so if pick 2-person group we get
$\boxed{\binom{5}{2} = 10}$

(d) Alice and Carol together reduces the problem to a 4 elt set $\{AC, B, D, E\}$.
Number of set partitions is $\boxed{B_4 = 15}$

If Bob and Don are always together too we get a 3 elt set $\{AC, BD, E\}$ and the number is $\boxed{B_3 = 5}$

So Bob and Don not together gives
$\boxed{B_4 - B_3 = 10}$

cont'd

#7 (d) These can also be explicitly written out.

(8) There are $2^{10}-1 = 1023$ possible nonempty subsets. The max possible ~~sum~~ sum is $a_1 + \cdots + a_{10}$, and since max is 20 and all are distinct the biggest this sum can be is $20+19+18+\cdots+11 = 155$. The smallest possible sum is the min val of the a_i, which is 1. So any possible sum S must satisfy

$$1 \leq S \leq 155.$$

Since $1023 > 155$ the pigeonhole principle means at least 2 subsets A, B with $A \neq B$ give the same sum. (Note: without restricting the max val of the a_i the result is false. e.g. what if $a_i = 10^i$?)

(9) Let E be the edges of G. We must compute

$$\sum_{\emptyset \subseteq F \subseteq E} (-1)^{|F|} \, n^{k(F)}$$

where $k(F)$ is the number of connected components of $G(F)$ (= graph with same verts as G, edges given by F).

$\lvert F \rvert$	possible $G(F)$ with multiplicity	$k(F)$	contribution
0	 $\times 1$	4	n^4
1	 $\times 4$	3	$-4n^3$
2	 $\times 2$,  $\times 4$	2	$+6n^2$
3	 $\times 4$	1	$-4n$
4	 $\times 1$	1	$+n$

$$\boxed{n^4 - 4n^3 + 6n^2 - 3n}$$