MATH 513 EXAM II

This exam is worth 100 points, with each problem worth 20 points. There are problems on *both sides* of the page. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must justify your answer to receive credit for a solution; correct answers alone are not necessarily sufficient for credit.

When submitting your exam, please indicate which problems (including Problem 1) you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly five problems; any unselected problems will not be graded, and if you select more than five only the first five (in numerical order) will be graded. You may use a calculator to assist with arithmetic. You can only use the basic functions (+, −, ×, /, =, memory) of the calculator; no advanced functions/programming is allowed. No phones, smartwatches, or other devices with connectivity can be used during the exam. The notation $[\![n]\!]$ means the finite set $\{1, \dots, n\}$. Unless requested, answers may be left in symbolic form.

(1) (20 pts) **True/False.** Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.
   (a) (4 pts) The factorials $a_n = n!$ satisfy the recurrence $a_0 = 1$, $a_n = na_{n-1}$ for $n \geq 1$.
   (b) (4 pts) The generating function $f(x) = 1 + x + x^2 + x^3 + \cdots$ can be written in closed form as $1/(1-x)$.
   (c) (4 pts) The Fibonacci numbers can be defined by the recurrence $f_0 = 1$, $f_1 = 2$, and $f_n = 2f_{n-1} + f_{n-2}$ for $n \geq 2$.
   (d) (4 pts) If $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$ are formal power series, then their product fg is $\sum_{n\geq 0} c_n x^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.
   (e) (4 pts) Let a_n be the number of permutations of $[\![n]\!]$. Then the exponential generating function for the numbers a_n is given by $1/(1-x)$.

(2) (20 pts) A sequence a_n, $n \geq 0$ satisfies $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$, and $a_0 = 1$, $a_1 = 3$.
   (a) (5 pts) Compute the sequence up to $n = 4$.
   (b) (15 pts) Determine an explicit formula for a_n that does not involve a sum.

(3) (20 pts) A sequence a_n, $n \geq 0$ satisfies $a_n = na_{n-1} - n(n-1)a_{n-2}$ for $n \geq 2$, and $a_0 = 1$, $a_1 = 1$.
   (a) (5 pts) Compute the sequence up to $n = 4$.
   (b) (15 pts) Determine an explicit formula for a_n that does not involve a sum.

(4) (20 pts) At the beginning of the year an artificial pond is built and stocked with 50 frogs. It is expected that each year the population of the frogs will increase to $4\times$ its size by the end of the year, and on the last day of each year 100 frogs are removed to move to a new pond. Let f_n be the population in the original pond after n years have elapsed. Thus $f_0 = 50$, $f_1 = 200 - 100 = 100$, $f_2 = 300$, and so on.
   (a) (6 pts) Write a recurrence relation satisfied by the frog population.
   (b) (7 pts) Give an explicit expression for the generating function $F(x) = \sum_{n\geq 0} f_n x^n$.
   (c) (7 pts) Give a closed formula for f_n.

Date: Thursday, 6 Nov 2025.

(5) (20 pts) Let $p_k(n)$ be the number of partitions of n into at most k parts. Find an expression for the generating function $\sum_{n\geq 0} p_k(n)x^n$.

(6) (20 pts) In the kingdom of Atrocia the unit of currency is called the trurl (T). The bills come in denominations of $1T$, $3T$, $7T$, and $15T$, and all bills are colored blue.

(a) (6 pts) Let t_n be the number of ways to make n trurls using these kinds of bills, drawn from an unlimited supply. We always put $t_0 = 1$. Write an expression for the generating function $G(x) = \sum_{n\geq 0} t_n x^n$.

(b) (7 pts) In a special anniversary year, the $3T$ bills get printed in blue, red, and purple, and the $7T$ bills get printed in blue and green. Write the generating function for t_n, assuming that color of the bills is taken into account (thus two blue $7T$ bills to make $14T$ should be counted as different from two green bills, or a blue and a green bill.)

(c) (7 pts) During the anniversary year there was a fire at the treasury and all blue $3T$ bills and all $15T$ bills were destroyed. Write the generating function for t_n now.

(7) (20 pts) We have an infinite supply of 1×1, 1×2, and 2×2 rectangles. Let a_n be the number of tilings of a $2\times n$ rectangle using these tiles, without overlaps. (The 1×2 rectangles can be rotated 90° for a tiling.) Put $a_0 = 1$.

(a) (6 pts) Compute a recurrence relation satisfied by the a_n. Check it by explicitly computing a_n for $n \leq 3$ by hand.

(b) (10 pts) Find a closed-form expression for the generating function $f(x) = \sum_{n\geq 0} a_n x^n$.

(c) (4 pts) Find an explicit expression for a_n that doesn't involve a sum.

(8) (20 pts) For $n \geq 0$ let Δ_n be the triangle in the plane with vertices $(0,0)$, $(n,0)$, $(0,n)$. Let a_n be the number of points in Δ and on its edges with integer coordinates. We put $a_0 = 1$, and we have $a_1 = 3$ and $a_2 = 6$ (Figure 1).

(a) (4 pts) Compute a_n for $n \leq 5$.

(b) (4 pts) Give a formula for a_n in terms of n that doesn't involve a sum (it is not necessary to prove your formula, as long as it is correct).

(c) (12 pts) Give a closed-form expression for the generating function $f(x) = \sum_{n\geq 0} a_n x^n$.

(9) (20 pts) Recall that the *derangements* are the sequence d_n, $n \geq 0$ that begins $1, 0, 1, 2, 9, 44, 265, \ldots$. It is known that they satisfy the recursion $d_n = nd_{n-1} + (-1)^n$. Let $f(x) = \sum_{n\geq 0} d_n x^n/n!$.

(a) (12 points) Write an explicit closed form expression for $f(x)$ in terms of functions you know. Your answer should not involve a sum.

(b) (8 points) Use your expression to derive a formula for d_n (your answer may involve a finite sum).

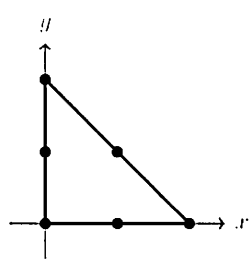

FIGURE 1.

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Answers

① ⓐ True ⓑ True ⓒ False (doesn't give the correct sequence) ⓓ True ⓔ True (The EGF is $\sum' a_n x^n/n!$ and $n! = a_n$ so we get $\sum' x^n = 1/(1-x)$).

② ⓐ $a_0 = 1$, $a_1 = 3$, $a_n = 3a_{n-1} - 2a_{n-2}$ $n \geq 2$
$a_2 = 9-2 = 7$, $a_3 = 21-6 = 15$,
$a_4 = 45 - 14 = 31$. (Looks like $a_n = 2^{n+1} - 1$)

ⓑ 2 solns: ① use the guess $a_n = 2^{n+1} - 1$ and use induction or ② use GFs.
Here is how ② goes. We know that the recurrence means $f(x) = \sum'_{n \geq 0} a_n x^n$ satisfies $f(x) = P(x)/(1-3x+2x^2)$
so

$$P(x) = (1-3x+2x^2)(1+3x+7x^2+15x^3+\cdots)$$
$$= \underbrace{1+0x}_{\text{these come from explicit computation}} + \underbrace{0x^2+0x^3+\cdots}_{\text{these are 0 because of the recurrence relation}}$$

so

$$f(x) = \frac{1}{1-3x+2x^2} = \underbrace{\frac{A}{1-2x} + \frac{B}{1-x}}_{\text{by factoring the denom.}}$$

(1)(b) (con't) so then

$$1 = (1-x)A + (1-2x)B$$

@ $x = \frac{1}{2}$ get $A = 2$
@ $x = 1$ get $B = -1$

so $\displaystyle\sum_{n\geq 0} a_n x^n = \frac{2}{1-2x} + \frac{-1}{1-x}$

$$= \sum_{n\geq 0} (2\cdot 2^n - 1)\, x^n$$

and $a_n = 2^{n+1} - 1$ ✓

(3) (a) $a_n = n a_{n-1} \overset{(*)}{-} n(n-1)\, a_{n-2} \quad n \geq 2$

$a_0 = a_1 = 1$

$a_2 = 2\cdot 1 - 2\cdot 1\cdot 1 = 0$

$a_3 = 3\cdot 0 - 3\cdot 2\cdot 1 = -6 \quad = -3!$

$a_4 = 4\cdot(-6) - 4\cdot 3\cdot 0 = -24 = -4!$

(a few more ...)

$a_5 = 5\cdot(-24) - 5\cdot 4\cdot(-6) = 0$

$a_6 = 6\cdot 0 - 6\cdot 5\cdot(-24) = 720 = 6!$

$a_7 = 7\cdot 720 - 7\cdot 6\cdot 0 = 5040 = 7!$

(b) again one can use induction or GFs. We show how to use GFs.

(*) This was a typo and was meant to be plus.

(2)

(3b) (cont) Put $f = \sum_{n \geq 0} a_n \frac{x^n}{n!}$ (EGF)

Then

$$f = 1 + x + \sum_{n \geq 2} a_n \frac{x^n}{n!}$$

$$= 1 + x + \sum_{n \geq 2} n a_{n-1} \frac{x^n}{n!} - \sum_{n \geq 2} n(n-1) a_{n-2} \frac{x^n}{n!}$$

$$= 1 + x + x \sum_{n \geq 1} a_n \frac{x^n}{n!} - x^2 \sum_{n \geq 0} a_n x^n/n!$$

$$= 1 + x + x(f-1) - x^2 f$$

so $f = 1/1 - x + x^2$ (**)

Now with the plus sign in the recurrence, we would get $-x^2$ in (**) and you would see the Fibonacci numbers. But alas we must continue.

write $1 - x + x^2 = (1 - \alpha x)(1 - \beta x)$ where α, β are the inverse roots. using QF the roots are

$$\frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

To find inverses we compute

$$\frac{2}{(1+\sqrt{-3})}\frac{(1-\sqrt{-3})}{(1-\sqrt{-3})} = \frac{2(1-\sqrt{-3})}{1+3} = \frac{1-\sqrt{-3}}{2}$$

and similarly

$$\frac{2}{(1-\sqrt{-3})} = \frac{1+\sqrt{-3}}{2} \quad \text{so} \quad \alpha = \beta^{-1}, \alpha^{-1} = \beta$$

we put $\alpha = (1+\sqrt{-3})/2$, $\beta = (1-\sqrt{-3})/2$.

Then we want

$$\frac{1}{1-x+x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \qquad \odot$$

so $1 = A(1-\beta x) + B(1-\alpha x)$

plug in $x = \beta^{-1} = \alpha$ and get

$$B = 1/1-\alpha^2$$

sim. $A = 1/1-\beta^2$.

Therefore we get using $\odot$

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} = \sum_{n\geq 0} (A\alpha^n + B\beta^n)\, x^n$$

and $a_n = n!\,(A\alpha^n + B\beta^n)$.

(4) $f_0 = 50,\ f_1 = 4\cdot 50 - 100 = 100$

$f_2 = 4\cdot 100 - 100 = 300$

$f_3 = 4\cdot 300 - 100 = 1100$

$f_4 = 4\cdot 1100 - 100 = 4300$ etc.

(a) The recurrence is $f_n = 4\cdot f_{n-1} - 100$, $n \geq 1$

$f_0 = 50$

(b) $f = \sum_{n\geq 0} f_n x^n = 50 + \sum_{n\geq 1} (4f_{n-1} - 100) x^n$

$= 50 + 4x \sum_{n\geq 0} f_n x^n - \sum_{n\geq 1} 100 x^n$

$= 50 + 4xf - 100x/1-x$

and $f = \dfrac{50}{1-4x} - \dfrac{100x}{(1-x)(1-4x)}$

(c) $50/1-4x = \sum_{n\geq 0} 50\cdot 4^n x^n$

$100x/1-x = \sum_{n\geq 1} 100\cdot x^n$

we ~~so~~ need $\dfrac{100x}{(1-x)(1-4x)} = \dfrac{A}{1-x} + \dfrac{B}{1-4x}$

4 (c) (cont'd)

$$100x = (1-4x)A + (1-x)B$$

$$\Longleftrightarrow \quad A+B=0 \quad \Rightarrow B=-A$$
$$-4A - B = 100$$
$$-4A + A = 100 \qquad A = -100/3$$
$$B = 100/3$$

so $f = \dfrac{50}{1-4x} + \left(\dfrac{-100/3}{1-x} + \dfrac{100/3}{1-4x} \right)$

$= \dfrac{50}{1-4x} + \dfrac{100/3}{1-x} - \dfrac{100/3}{1-4x}$

and $a_n = 50 \cdot 4^n + \dfrac{100}{3}\left(1 - 4^n\right)$

$n \geq 0.$

5 $p_k(n)$ = # of partitions of n into at most k parts.

If a partition has at most k parts, then its conjugate has largest part <u>at</u> <u>most</u> <u>k</u>

e.g. $k=4$

at most 4 parts

largest part is at most 4

5 (cont'd)

we know that we can write the OGF for partitions as

$$\sum_{n\geq 0}' p(n)\,x^n = \prod_{\ell=1}^{\infty} \frac{1}{1-x^\ell}$$

and the factor $\dfrac{1}{1-x^\ell} = 1+x^\ell+x^{2\ell}+\cdots$

gives the contribution of the part ℓ. So if we want at most k parts we must take the product with $\ell \leq k$:

$$\sum_{n\geq 0}' p_k(n)\,x^n = \prod_{\ell=1}^{k} \frac{1}{1-x^\ell}.$$

6 (a) This is the OGF for partitions where the only allowed parts are 1, 3, 7, 15. So we get

$$\sum_{n\geq 0} t_n x^n = \frac{1}{(1-x)(1-x^3)(1-x^7)(1-x^{15})}$$

(b) If we have m different types of parts of size l, then the factor in the GF becomes

$$\frac{1}{(1-x^l)^m}.$$

Therefore the GF is now

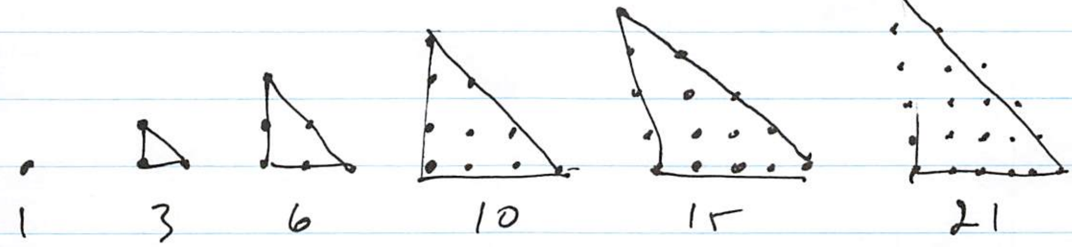
$$\frac{1}{1-x}\cdot\frac{1}{(1-x^3)^3}\cdot\frac{1}{(1-x^7)^2}\cdot\frac{1}{(1-x^{15})}$$

(c) Now we have no 15T bills and only 2 types of 3T bills, so we get

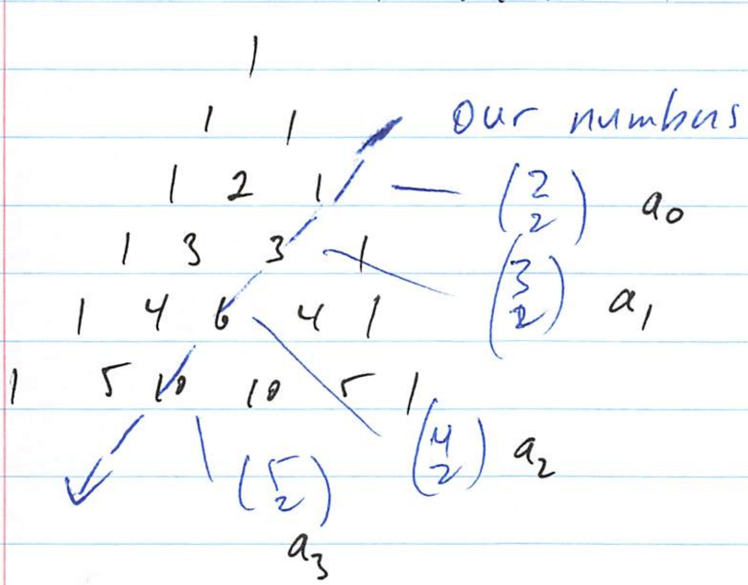
$$\frac{1}{1-x}\cdot\frac{1}{(1-x^3)^2}\cdot\frac{1}{(1-x^7)^2}$$

(7) was omitted from the exam

(8) (a) The triangles look like



⑧ (b) It is clear that we are getting the binomial coeffs $\binom{m}{2}$, although we have to make the indexing work.



$\Longrightarrow \quad a_n = \binom{n+2}{2}$

so our OGF is $f = \sum_{n \geq 0} \binom{n+2}{2} x^n$

(c) Write f as $\frac{1}{2} \sum_{n \geq 0} (n+2)(n+1)\, x^n$

now $(n+2)(n+1)\, x^n = \frac{d^2}{dx^2}\left(x^{n+2}\right)$

so $f = \frac{1}{2} \left(\frac{d}{dx}\right)^2 \sum_{n \geq 0} x^n$

and since $\sum_{n \geq 0} x^n = \frac{1}{1-x}$

8 c (cont'd) we get

$$f = \frac{1}{2}\left(\frac{d}{dx}\right)^2 \frac{1}{1-x}$$

$$= \frac{1}{2}\left(\frac{d}{dx}\right) \frac{1}{(1-x)^2}$$

$$= 1/(1-x)^3$$

9 a

$$f = \sum_{n \geq 0} d_n x^n/n!$$

$$= 1 + \sum_{n \geq 1} d_n x^n/n!$$

$$= 1 + \sum_{n \geq 1} \left(n d_{n-1} + (-1)^n\right) \frac{x^n}{n!}$$

$$= 1 + x \sum_{n \geq 1} d_{n-1} \frac{x^{n-1}}{(n-1)!} + \sum_{n \geq 1} \frac{(-x)^n}{n!}$$

$$= x f + 1 + \sum_{n \geq 1} (-x)^n/n!$$

$$= x f + e^{-x}$$

$$\Longrightarrow f = e^{-x}/1-x$$

9 (9)

$$\sum_{n\ge 0} d_n \frac{x^n}{n!} = \left(\sum_{n\ge 0} (-1)^n \frac{x^n}{n!}\right)\left(\sum_{n\ge 0} x^n\right)$$

so $$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \cdot 1$$

and $$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

This is the formula we got before using Inclusion/Exclusion.