MATH 513 EXAM II (TAKE HOME)

This exam is worth 100 points, with each problem worth 20 points. Please complete *any five* of the problems. You must justify your answer to receive credit for a solution; correct answers alone are not necessarily sufficient for credit.

The exam will be submitted on Moodle as individual problems (like with the HW). Please submit exactly five problems; if you submit more than five only the first five (in numerical order) will be graded. Please make sure your name and student ID are written somewhere in your answers. Also please name your PDF files in the form

`LastName_FirstName_Student_ID_Exam2_ProblemXX.pdf`

ADDITIONAL INSTRUCTIONS FOR TAKE-HOME EXAM.

The exam answers must be submitted in PDF, just like with HW submissions. Scans of handwriting are ok, but please be sure that they are at a sufficiently high resolution for me to be able to read them. The following **are allowed**:

- You may use class materials (textbook, your own notes, hw assignments including solutions, lecture notes from video lectures, and video lectures) during the exam.
- You may use the Desmos Scientific calculator `https://www.desmos.com/scientific` to assist with numerical computations. You may also use your own calculator if you prefer it; Desmos is allowed so that everyone is guaranteed to have access to something.

The following **are not allowed**:

- Discussing the exam with anyone in the class or elsewhere. Exception: you may ask me by email for clarification about a problem, just like in the classroom exam. I will try to check email often but unavoidably there will be delays in replies.
- Using any other sources of information (internet, other books, other notes, tables, Wikipedia, etc.) during the exam.
- Using a computer (other than Desmos above or for access to video lectures and our course page). In particular programming is not allowed.

When submitting your exam, you are agreeing to the following statement:

I hereby declare that the work submitted represents my individual effort. I have neither given nor received any help and have not consulted any online resources. I attest that I have followed the instructions of the exam.

Date: Assigned: Friday 17 Apr 2020; Due: on Moodle by 12 noon, Monday 20 Apr 2020.

(1) (20 pts) A college registrar reports the following information about a group of 400 students. There are 180 taking a math class, 200 taking an English class, 160 taking a biology class, and 250 in a foreign language class. 80 are enrolled in both math and English, 90 in math and biology, 120 in math and a foreign language, 70 in English and biology, 140 in English and a foreign language, and 60 in biology and a foreign language. Also, there are 25 in math, English, and a foreign language, 30 in math, English, and biology, 40 in math, biology, and a foreign language, and fifteen in English, biology, and a foreign language. Finally, the sum of the number of students with a course in all four subjects, plus the number of students with a course in none of the four subjects, is 100. Determine the number of students that are enrolled in all four subjects simultaneously: math, biology, English, and a foreign language.

(2) (20 pts) Compute the number of odd positive integers $n \leq 10000$ that are not divisible by 7 and 11.

(3) (20 pts) Let $t(n)$ be the number of partitions of n where each part is divisible by 3. Put $t(0) = 1$.
   (a) (6 pts) Compute $t(n)$ for $n \leq 10$.
   (b) (7 pts) Show that $t(n)$ also equals the number of partitions of n where the number of times each part occurs is divisible by 3.
   (c) (7 pts) Give a formula for $t(n)$ using the ordinary partition function.

(4) (20 pts) Use the technique of generating functions to solve the recurrence relation

$$a_n = na_{n-1} + n!, \quad a_0 = 1$$

and determine an explicit formula for a_n. As part of your answer, please compute a_n, $n \leq 3$ directly using the recurrence relation, and then verify that your formula for a_n reproduces those values.

(5) (20 pts) Use the technique of generating functions to solve the recurrence relation

$$a_n = (n+1)a_{n-1} + 2^{n+1}, \quad a_0 = 2$$

and determine an explicit formula for a_n. As part of your answer, please compute a_n, $n \leq 3$ directly using the recurrence relation, and then verify that your formula for a_n reproduces those values.

(6) (20 pts) In Figure 1, a beam of light is passing through two adjacent panes of glass. The beam can be reflected or it can pass through the glass. Let a_n be the number of paths the beam can take if there are n reflections. Figure 1 shows that $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$. (We do not count the reflection that bounces the beam off the top edge; the beam must enter the glass.)
   (a) (6 pts) Determine a_3, a_4.
   (b) (7 pts) Write a recurrence relation satisfied by the a_n.
   (c) (7 pts) Solve the recurrence relation to express the generating function $\sum_{n \geq 0} a_n x^n$ as a rational function.

(7) (20 pts) Consider a $1 \times n$ rectangle R. We have an unlimited supply of 1×2 tiles and 1×3 tiles. The 1×2 tiles come in two different colors (red and green) and the 1×3 tiles come in four different colors (black, yellow, purple, and orange). Let a_n be the number of ways to tile R using the 1×2 and 1×3 tiles. Put $a_0 = 1$.

- (a) (6 pts) Compute a_3 and a_4.
- (b) (7 pts) Find a recurrence relation satisfied by the a_n and use it to determine the generating function $\sum_{n \geq 0} a_n x^n$ as a rational function.
- (c) (7 pts) Show that a_n is approximately $\frac{2}{5} \cdot 2^n$ when n is large. Hint: this answer tells you something about the denominator of the rational function from the 2nd part.

(8) Let a_n be the number of strings in the symbols A, B, C of length n that do not contain BC or CB as a length 2 substring. For instance $a_1 = 3$ and $a_2 = 7$ because of the strings

$$AA, BB, CC, AC, AB, BA, CA.$$

- (a) (6 pts) Compute a_3.
- (b) (7 pts) Find a recurrence relation of order 2 satisfied by the a_n.
- (c) (7 pts) Find an exact formula for a_n. Hint: your answer will involve $\sqrt{2}$.

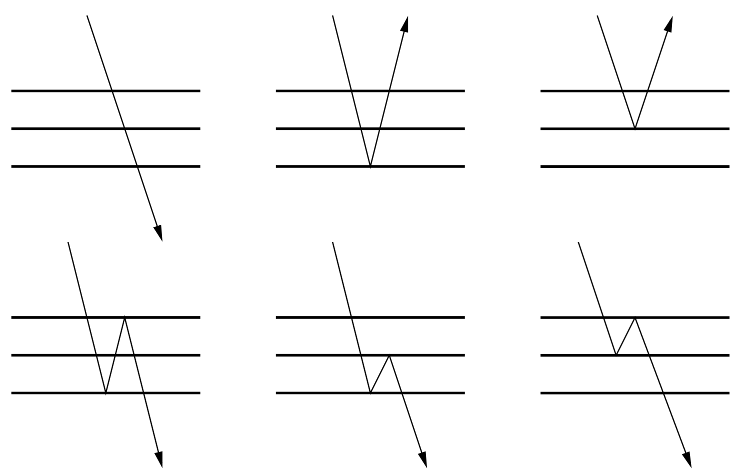

FIGURE 1.