MATH 513 FINAL EXAM (TAKE HOME)

This exam is worth 100 points, with each problem worth 20 points. Please complete *any five* of the problems. You must justify your answer to receive credit for a solution; correct answers alone are not necessarily sufficient for credit.

The exam will be submitted on Moodle as individual problems (like with the HW). Please submit exactly five problems; if you submit more than five only the first five (in numerical order) will be graded. Please make sure your name and student ID are written somewhere in your answers. Also please name your PDF files in the form

`LastName_FirstName_Student_ID_FinalExam_ProblemXX.pdf`

ADDITIONAL INSTRUCTIONS FOR TAKE-HOME EXAM.

The exam answers must be submitted in PDF, just like with HW submissions. Scans of handwriting are ok, but please be sure that they are at a sufficiently high resolution for me to be able to read them. The following **are allowed**:

- You may use class materials (textbook, your own notes, hw assignments including solutions, lecture notes from video lectures, and video lectures) during the exam.
- You may use the Desmos Scientific calculator `https://www.desmos.com/scientific` to assist with numerical computations. You may also use your own calculator if you prefer it; Desmos is allowed so that everyone is guaranteed to have access to something.
- You may use the pari-gp online calculutor `https://pari.math.u-bordeaux.fr/gp.html` if you like. It could be useful for checking your work with generating functions, like we did in class. See below for restrictions.

The following **are not allowed**:

- Discussing the exam with anyone in the class or elsewhere. Exception: you may ask me by email for clarification about a problem, just like in the classroom exam. I will try to check email often but unavoidably there will be delays in replies.
- Using any other sources of information (internet, other books, other notes, tables, Wikipedia, etc.) during the exam.
- Using a computer (other than that mentioned above or for access to video lectures and our course page). In particular programming is not allowed.

When submitting your exam, you are agreeing to the following statement:

I hereby declare that the work submitted represents my individual effort. I have neither given nor received any help and have not consulted any online resources. I attest that I have followed the instructions of the exam.

Date: Assigned: Friday 1 May 2020; Due: on Moodle by 11:59 PM EDT, Thursday 7 May 2020.

(1) (20 pts) According to the *Laws of the Game of the International Football Association*, a full football (soccer) team consists of eleven players, one of whom is the goalkeeper. The other ten players fall into one of three outfield positions: defender, midfielder, and striker. There is no restriction on the number of players at each of these positions, as long as the total number of outfield players is ten.

   (a) (6 pts) How many different configurations are there for a full football team? For example, one team may field four strikers, three midfielders, and three defenders, in addition to the goalkeeper. Another may play five strikers, no midfielders, and five defenders, plus the goalkeeper.

   (b) (7 pts) Repeat the previous problem if there must be at least two players at each outfield position.

   (c) (7 pts) How many ways can a coach assign eleven different players to one of the four positions, if there must be exactly one goalkeeper, but there is no restriction on the number of players at each outfield position?

(2) (20 pts) On the island of Atrocia, all cars have license plates consisting of six numerical digits only. The digits are `0`–`9` and `0` can be a leading digit.

   (a) (6 pts) How many license plates are there?

   (b) (14 pts) A witness to a crime could only give a partial description of the getaway car. In particular, she noticed that the license plate was from Atrocia, there was only one digit that occurred more than once, and that digit occurred three times. How many license plates have this property?

(3) (20 pts) Let $k_1 + \cdots + k_m = n$. Recall that $\binom{n}{k_1, k_2, \ldots, k_m}$ denotes the multinomial coefficient $\frac{n!}{k_1! k_2! \cdots k_m!}$. In the following you may use any proof technique you like (combinatorial or otherwise).

   (a) (6 pts) Prove

$$\sum_{k_1+\cdots+k_m=n} \binom{n}{k_1, \ldots, k_m} = m^n.$$

   Hint: Multinomial theorem (§1.10, exercise 29.)

   (b) (7 pts) Prove

$$\binom{n}{k_1, \ldots, k_m} = \sum_{i=1}^{m} \binom{n-1}{k_1, \ldots, k_i - 1, \ldots, k_m}.$$

   (c) (7 pts) Suppose k is fixed. Prove

$$\sum_j \binom{n}{j, k, n-j-k} = 2^{n-k} \binom{n}{k}.$$

(4) (20 pts) To celebrate the end of the semester, students decide to have a flag party. They have an unlimited supply of square flags that are k feet on a side for k any positive integer. Each flag comes in one of four different colors. Each k foot by k foot flag has a picture of a labeled tree of order k with vertices labeled by $\{1, \ldots, k\}$. Let a_n be the number of ways to arrange flags of total side length n on an n foot flagpole. Put $a_0 = 1$.

   (a) (4 pts) Check that $a_1 = 4$ and $a_2 = 20$.

(b) (4 pts) Compute a_3.
(c) (8 pts) Use generating functions to explain how to compute the sequence a_n. (It is not necessary to give an explicit formula for a_n.)
(d) (4 pts) Use your answer to verify your computation of a_3.

(5) (20 pts) A teacher has n students and wants to assign projects to them. She divides them arbitrarily into two groups A and B (a given group may be empty). The students in group A each take a practice test in one of four subjects. The students in group B arrange themselves around a circular table to have a group discussion about one of three different topics. (She considers two circular arrangements to be the same if each student has the same person sitting on his/her left in the two arrangements.) Let a_n be the number of ways to do this, and put $a_0=1$.
(a) (6 pts) Compute a_1, a_2, and a_3.
(b) (7 pts) Use generating functions to explain how to compute the a_n.
(c) (7 pts) Use your generating function to give an explicit formula for a_n. Verify your computation of a_3.

(6) (20 pts) Let $\mathcal{T}_n$ be the set of *unlabeled trees* on n vertices, where $n \geq 2$. The sequence $|\mathcal{T}_n|$, $n \geq 2$ goes

$$1, 1, 2, 3, 6, 11, 23, 47, 106, 235, \ldots$$

(a) (8 pts) Check explicitly (by drawing the trees and counting their automorphisms) that the formula

$$\sum_{T \in \mathcal{T}_n} \frac{1}{|\mathrm{Aut}(T)|} = \frac{n^{n-2}}{n!}. \tag{1}$$

is true for $n = 2, 3, 4, 5$.
(b) (12 pts) Prove that the formula (1) holds for all n. *Hint:* group labeled trees together if they have the same underlying unlabeled tree.

(7) (20 pts) Let $n \geq 3$ and let $2 \leq k \leq n-1$. Define a graph $G(k, n)$ as follows. The vertices of $G(k, n)$ are the subsets of $\{1, \ldots, n\}$ of order k. Two vertices I, J are joined by an edge if and only if $|I \cap J| = k - 1$.
(a) (5 pts) Draw pictures of $G(2, 4)$ and $G(3, 5)$.
(b) (5 pts) Prove that $G(k, n)$ is connected.
(c) (5 pts) The graph $G(k, n)$ is *regular*, i.e. each vertex has the same degree (you don't have to prove that). Compute the degrees of the vertices of $G(k, n)$.
(d) (5 pts) Compute the number of edges of $G(k, n)$.

(8) (20 pts) Let π_1 and π_2 be two set partitions of $\{1, \ldots, n\}$, where $n \geq 2$. Let us say π_1 is a *refinement* of π_2 if every block in π_2 is a union of blocks in π_1. For example, $\{\{1, 2\}, \{3, 4\}, \{5, 6, 7\}\}$ is a refinement of $\pi = \{\{1, 2, 3, 4\}, \{5, 6, 7\}\}$, but $\{\{1, 2\}, \{3, 4, 5\}, \{6, 7\}\}$ is not a refinement of π.

Now define a graph P_n as follows. The vertices are all the set partitions of $\{1, \ldots, n\}$. Two vertices π, π' are joined by an edge if and only if either π is a refinement of π' or π' is a refinement of π. Figure 1 shows the graph P_3.
(a) (6 pts) Draw a picture of P_4.
(b) (7 pts) Prove that P_n is connected for $n \geq 4$.

(c) (7 pts) Let Q_n be the subgraph of P_n where we delete the vertices corresponding to the two set partitions with 1 part and n parts, as well as all the edges that are incident to these vertices. For instance, Q_3 is obtained from P_3 by deleting the top and the bottom vertices, as well as all the edges. The example of Q_3 shows that this graph is not necessarily connected. Is Q_n connected for $n \geq 4$? Why or why not?

(9) (20 pts) Let $X = \{1, 2, \ldots, n\}$ for $n \geq 1$. Let Y be the positive integers $\{1, 2, 3, \ldots\}$. Let us say a function $f\colon X \to Y$ is *shrinking* if there exists a $k \geq 1$ such that f is *onto* the subset $\{1, 2, \ldots, k\} \subset Y$. Let S_n be the number of shrinking functions with domain X. Put $S_0 = 1$. Then $S_1 = 1$, corresponding to the function 1 gets taken to 1, and $S_2 = 3$, corresponding to the three functions (i) both 1 and 2 get taken to 1, (ii) 1 gets taken to 1 and 2 gets taken to 2, and (iii) 1 gets taken to 2 and 2 gets taken to 1.

(a) (8 pts) Compute S_3 by exhibiting all the possible shrinking functions f with domain $\{1, 2, 3\}$.

(b) (12 pts) Consider the exponential generating function $A(x) = \sum_{n \leq 0} S_n x^n / n!$. Show that
$$A(x) = \frac{1}{2 - e^x}.$$

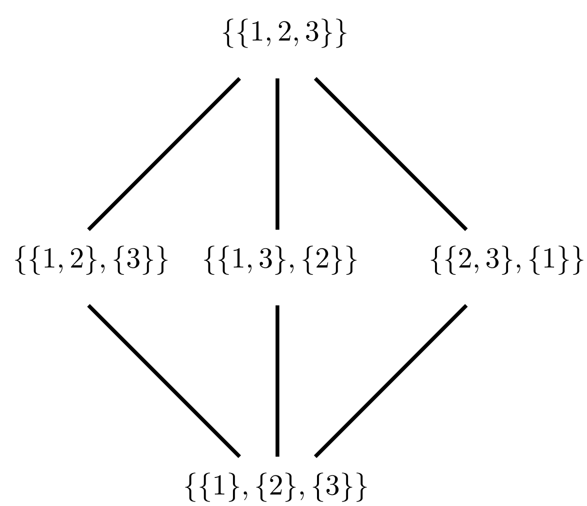

Figure 1. The graph P_3